Fitting Correlations within and between Bond Markets.

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Abstract

In this paper we estimate and test a multi-factor CIR model for three countries: the USA, Germany and the UK. We find that the estimated model reproduces not only the correlation within each of the bond markets considered but also those observed between markets, suggesting the existence of common factors.

Keywords: multi-factor CIR model, Kalman filter, common factors.

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1 Introduction

In this paper we estimate and test a multi-factor CIR model, which is exponential-affine in the state variables, for three countries: the USA, Germany and the UK. The goal of this cross-country estimation and testing is twofold. First, although the literature on empirical tests of the CIR model (either single or multi-factor) is vast, most empirical tests have been performed on US data only. To our knowledge there is no paper that estimates this model for different countries over an identical sample period with the proposed methodology. By bringing together the empirical findings for these three countries we are able to compare the outcome of this model in terms of factor dynamics, risk premia and factor composition. Second, we want to determine to what extent multi-factor models can be useful in explaining (fitting) correlation not only within but also between national bond markets. A positive answer to this question is of obvious relevance for developing parsimonious international multi-factor models for the yield curves with obvious applications in international finance. Some of the applications of arbitrage free pricing in international finance can for instance be found in Backus, Foresi and Telmer (1998) and Ahn (1997).

The strong cross-sectional correlation between bond yields of differing maturities has inspired various researchers to find and determine a limited set of factors that may drive the entire yield curve of a given (national) bond market. One popular route in the finance literature to find this set of factors as well as its dynamics is to assume some diffusion process for the short rate and then use arbitrage arguments to find the functional form of and relations between observed yields of bonds with varying maturities. Seminal examples of this approach include Cox, Ingersoll and Ross (1985, CIR hereafter) and Chen and Scott (1993). Even within this framework, however, closed form solutions for bond prices (and yields) are hard to obtain without rather strong assumptions on the diffusions of the short rate process. One class of diffusions for which closed form solutions exist is the class of affine term structure models, see Duffie and Kan (1996). This class embeds as special cases the Vasicek (1977), CIR (1985), Chen and Scott (1993) and Hull and White (1990) models.

We follow the recent literature on multi-factor exponential-affine term structure model estimation (Duan and Simonato (1998), de Jong (1997) and Geyer and Pichler (1998)). More in particular we use a Kalman filter approach to filter the actual dynamics for the latent factors. This methodology, as is well known by now, has the advantage over other alternatives in that it efficiently combines the time series and the cross-sectional dimension of the yield curve. Traditional empirical models of interest rates focused either on fitting a cross-section of bond prices (e.g. Brown and Dybvig (1986)) or on fitting a time-series model to one particular maturity (e.g. Chan, Karolyi, Longstaff and Sanders (1992)). Moreover, the Kalman filter approach has the advantage that no *ad hoc* assumptions have to be made about the measurement error structure, which is not the case for some alternative filtering methods (e.g.

inverting approach of Pearson and Sun (1994) and Chen and Scott (1993)). Although the Kalman filter does not represent the optimal filtering methodology (some of its assumptions do not hold true in the exponential-affine term structure setup) it can still be interpreted as quasi-optimal filtering. Biases tend to be small for the relevant parameter ranges and sample sizes (Duan and Simonato (1998), de Jong (1997) and Bollerslev and Woolridge (1992)).
The remainder of the paper is organized in five sections. Section 2 describes the multi-factor CIR model, section 3 describes the state space representation of the exponential-affine term structure model and explains the Kalman filter procedure and the econometric properties of the estimator. Section 4 presents the estimation results along with some diagnostic checks and subsequently verifies whether the specific multi-factor model can replicate the observed correlation within and between bond markets of the three countries considered. Finally, section 5 concludes.

2 The multi-factor CIR model for bond prices and yields

The set-up is standard and widely understood (for a more extensive discussion, see e.g. Duffie (1996) and Duffie and Kan (1996)). In short, we fix a standard Brownian motion $\mathbf{W(t)} = [W_1(t), ..., W_K(t)]'$ in $\mathbb{R}^K$ ($K \geq 1$), restricted to a given time interval $[0, T]$ on a given[1] complete probability space $(\Omega, \mathcal{F}, P)$. Implicitly we assume that bond markets are frictionless, that investors are insatiable and that arbitrage opportunities are absent in the market. Moreover we assume that $dW_i(t)dW_j(t) = 0$ for all $i = \{1, ..., K\}$ and $j \neq i$. We also fix the information structure as the standard filtration $\mathbb{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}$ of $\mathbf{W(t)}$.
We consider K factors or state variables, $\mathbf{X(t)} = [X_1(t), ..., X_K(t)]'$, describing bond prices in the economy. Each factor $X_i(t)$, $i \in \{1, .., K\}$, is the solution of a (Markovian) CIR stochastic differential equation :

$$dX_i\left(t\right) = \kappa_i\left(\theta_i - X_i(t)\right)dt + \sigma_i\sqrt{X_i(t)}dW_i\left(t\right). \tag{1}$$

The dimensions of the parameters are standard. The instantaneous riskless return is defined as

$$r_t \equiv \sum_{i=1}^{K} X_i(t), \tag{2}$$

generating what is known as a multi-factor CIR model. The CIR model assumes moreover that the market price of risk for each state variable is proportional to its standard deviation :

$$\lambda_i(X_i(t), t) = \frac{\lambda_i}{\sigma_i}\sqrt{X_i(t)}. \tag{3}$$

[1] It is nontrivial to show that one can always construct a probability space so that there exist standard Brownian motions.

Under these assumptions and in the absence of arbitrage, merely technical conditions are required for the existence of an equivalent martingale measure (see Duffie (1996, pp.110-111)). Such a probability measure Q has the property that any security with certain payoff 1 at time of maturity $T = t + \tau \geq t$ has a price $P_t(\tau)$ at any time $t \leq T$ of:

$$P_t(\tau) \equiv E_t^Q \left[\exp\left(-\int_t^{t+\tau} r_u du \right) \cdot 1 \right] = f(X_1(t), ..., X_K(t), \tau), \tag{4}$$

where E_t^Q denotes $\mathcal{F}_t$-conditional expectation under Q. The above process $P_t(\tau)$ as a function of the time to maturity τ is denoted as the term structure of interest rates at time t. Given the above general representation of the solution (4), the Feynman-Kac formula provides us with the (fundamental) partial differential equation that corresponds to this solution. Loosely put, Feynman and Kac prove that $E_t^Q \left[\frac{dP_t(\tau)}{P_t(\tau)} \right] = r_t$, or more formally :

$$\mathcal{D}^Q f(X_1(t), ..., X_K(t), \tau) - r_t f(X_1(t), ..., X_K(t), \tau) = 0, \tag{5}$$

with boundary condition $f(r_T, T) = 1$, where $\mathcal{D}$ is the Dynkyn operator and in this case stands for :

$$\mathcal{D} f(X_1(t), ..., X_K(t), \tau) = \frac{\partial f}{\partial t} + \sum_{i=1}^{K} \frac{\partial f}{\partial X_i} \mu_i(t) + \frac{1}{2} \sum_{i=1}^{K} \frac{\partial^2 f}{\partial X_i^2} \sigma_i^2 X_i(t), \tag{6}$$

where the ith factor drift under Q, $\mu_i(t)$, $i = \{1, ..., K\}$, is defined as[2]:

$$\mu_i(t) = \kappa_i(\theta_i - X_i(t)) - \lambda_i X_i(t). \tag{7}$$

In general closed form solutions for the bond prices, i.e. functions f solving (5), are not available. By restricting the instantaneous drift and variance functions (under Q) and r_t to be affine in the state variables[3], as is the case in the present CIR multi-factor setting, one can solve (5) quite easily by solving a system of $2 \cdot K$ ordinary differential equations. In CIR (1985) it is proven that the following arbitrage free, closed form solution can be obtained :

$$P_t(\tau; \vartheta) = \prod_{i=1}^{K} \exp\left(A_i(\tau; \vartheta_i) - B_i(\tau; \vartheta_i) X_i(t) \right), \tag{8}$$

where ϑ_i is the set of parameters corresponding to factor i, $\vartheta_i = [\kappa_i, \theta_i, \sigma_i, \lambda_i]'$, and where

$$A_i(\tau; \vartheta_i) = ln \left(\frac{2\phi_{i,1} \exp(\phi_{i,2} \cdot \frac{\tau}{2})}{\phi_{i,4}} \right)^{\phi_{i,3}} \tag{9}$$

$$B_i(\tau; \vartheta_i) = \frac{2(\exp(\phi_{i,1} \cdot \tau) - 1)}{\phi_{i,4}} \tag{10}$$

[2] Note that under Q the dynamics of the factors behave as:

$$dX_i(t) = r_i(t)dt + \sigma_i \sqrt{X_i(t)} d\tilde{W}_i(t),$$

where $d\tilde{W}_i(t)$ is obtained from $dW_i(t)$ via the Girsanov theorem, $d\tilde{W}_i(t) = dW_i(t) + \lambda_i(X_i(t), t)$.

[3] A function $f_t(x)$ is called affine (in x) if there exist constants α and β such that for all x, $f_t(x) = \alpha + \beta x$.

are continuously differentiable functions of τ and $\phi_{i,1} = \sqrt{(\kappa_i + \lambda_i)^2 + 2\sigma_i^2}$, $\phi_{i,2} = \kappa_i + \lambda_i + \phi_{i,1}$, $\phi_{i,3} = 2\kappa_i\theta_i/\sigma_i^2$, and $\phi_{i,4} = 2\phi_{i,1} + \phi_{i,2}(\exp(\phi_{i,1} \cdot \tau) - 1)$.

Finally, note that the risk premium for a bond can be easily recovered from the estimated parameters. Indeed, under the risk neutral probability we have that the instantaneous expected return on a bond is given by:

$$\frac{\mathcal{D}^Q f(X_1(t), ..., X_K(t), \tau)}{f(X_1(t), ..., X_K(t), \tau)} = r_t. \tag{11}$$

The instantaneous expected change in price under Q, $\mathcal{D}^Q f(X_1(t), ..., X_K(t), \tau)$ relates to the instantaneous expected change in price under the historical measure P, $\mathcal{D}^P f(X_1(t), ..., X_K(t), \tau)$, as:

$$\mathcal{D}^Q f(X_1(t), ..., X_K(t), \tau) = \mathcal{D}^P f(X_1(t), ..., X_K(t), \tau) - \sum_{i=1}^{K} \frac{\partial f}{\partial X_i} \lambda_i X_i(t), \tag{12}$$

such that the expected rate of return under P equals:

$$\begin{aligned} \frac{\mathcal{D}^P f(X_1(t), ..., X_K(t), \tau)}{f(X_1(t), ..., X_K(t), \tau)} &= r_t + \sum_{i=1}^{K} \frac{\partial \ln(f)}{\partial X_i} \lambda_i X_i(t) \\ &= r_t - \sum_{i=1}^{K} B_i(\tau) \lambda_i X_i(t). \end{aligned} \tag{13}$$

Note the double role of the dynamics of the factors from equation (13). First, the dynamics determine the cross-sectional expected returns through the functions $B_i(\cdot)$. Second, the dynamics through time of expected returns will depend on the value the factors take at each point in time. The value of each of the factors will both determine the level of the instantaneous interest rate ($r_t \equiv \sum_{i=1}^{K} X_i(t)$) as well as the risk premium of each of the bonds. Both dimensions thus depend on the specification of the dynamics of the factors.

3 Empirical implementation

We apply the standard Kalman-filter approach for the implementation of the exponential-affine multi-factor CIR model, as in Duan and Simonato (1998), Geyer and Pichler (1998) and de Jong (1997). Our aim is to filter the unobserved state variables and to estimate their dynamics. As a by-product we obtain the model-predicted yield curve.

3.1 State space representation

Filtering the unobserved state variables usually calls for either assuming no sampling error (such that the yield curve can be inverted at certain points to deliver the state variables), or the use of approximate filtering techniques, such as the Kalman filter. The first approach has been followed by, amongst others, Pearson and Sun (1994) and Chen and Scott (1993). This approach implies a discretionary choice of maturities used to invert the yield curve and to extract the implied state variables. Obviously, the choice of maturities used in the inversion procedure is not innocuous and the results are thus potentially very sensitive to the particular choice of maturities made. The second approach, followed here, is to assume that all the yields are measured with error. Hence standard inversion procedures no longer work and one has to rely on some filtering procedure to filter the state variables from the yield curve. The Kalman filter approach turns out to be an approximately 'correct' filtering technique.

In the remainder of this paper, we will work with the continuously compounded yield (or yield curve) notation observed for a set of N different bond prices with accordingly defined maturity set $\boldsymbol{\tau} = [\tau_1, ..., \tau_N]'$, the link between the two being ($j = \{1, ..., N\}$):

$$\begin{aligned} P_t(\tau_j) &\equiv \exp(-\tau_j \cdot Y_t(\tau_j)) \cdot 1 \\ &\Leftrightarrow \\ Y_t(\tau_j) &\equiv -\frac{ln(P_t(\tau_j))}{\tau_j}. \end{aligned} \tag{14}$$

Denote the $(N \times 1)$ vector of yields to maturity as observed at time t as $\mathbf{Y_t}(\boldsymbol{\tau}) = [Y_t(\tau_1), ..., Y_t(\tau_N)]'$. Define the state-space vector at time t as $\mathbf{X(t)} = [X_1(t), ..., X_K(t)]'$, then the state space representation for every yield is given by :

$$\mathbf{Y_t}(\boldsymbol{\tau}) = \mathbf{AA}(\boldsymbol{\tau};\boldsymbol{\vartheta}) + \mathbf{BB}(\boldsymbol{\tau};\boldsymbol{\vartheta})\,\mathbf{X(t)} + \boldsymbol{\xi}_\mathbf{t}(\boldsymbol{\tau};\boldsymbol{\vartheta}), \tag{15}$$

where $\mathbf{AA}(\boldsymbol{\tau};\boldsymbol{\vartheta})$ and $\mathbf{BB}(\boldsymbol{\tau};\boldsymbol{\vartheta})$ are $(N \times 1)$ and $(N \times K)$ matrices, defined as:

$$\mathbf{AA}(\boldsymbol{\tau};\boldsymbol{\vartheta}) = \begin{bmatrix} a(\tau_1;\boldsymbol{\vartheta}) \\ \vdots \\ a(\tau_N;\boldsymbol{\vartheta}) \end{bmatrix} \text{ and } \mathbf{BB}(\boldsymbol{\tau};\boldsymbol{\vartheta}) = \begin{bmatrix} b_1(\tau_1;\boldsymbol{\vartheta}) & \cdots & b_K(\tau_1;\boldsymbol{\vartheta}) \\ \vdots & \cdots & \vdots \\ b_1(\tau_N;\boldsymbol{\vartheta}) & \cdots & b_K(\tau_N;\boldsymbol{\vartheta}) \end{bmatrix}, \tag{16}$$

where $a(\tau_j;\boldsymbol{\vartheta}) = -\frac{1}{\tau_j}\sum_{i=1}^{K} A_i(\tau_j;\vartheta_i)$, for all maturities $j = \{1, ..., N\}$ in the dataset, and $b_i(\tau_j;\boldsymbol{\vartheta}) = \frac{1}{\tau_j}B_i(\tau_j;\vartheta_i)$, for all maturities $j = \{1, ..., N\}$ in the dataset and for all state variables $i = \{1, ..., K\}$. The observation errors as observed at time t are gathered in the $(N \times 1)$ vector $\boldsymbol{\xi}_\mathbf{t}(\boldsymbol{\tau};\boldsymbol{\vartheta})$. For simplicity, we assume that these errors are cross-sectionally and intertemporally uncorrelated with $E\left(\boldsymbol{\xi}_\mathbf{t}(\boldsymbol{\tau};\boldsymbol{\vartheta})\right) = \mathbf{0}$ and $E\left(\boldsymbol{\xi}_\mathbf{t}(\boldsymbol{\tau};\boldsymbol{\vartheta})\boldsymbol{\xi}_\mathbf{t}(\boldsymbol{\tau};\boldsymbol{\vartheta})'\right) = \mathbf{H}$. Equation (15) is called the measurement equation and models the cross-sectional variation in the yields.

The intertemporal dimension of the data is modelled through the transition equation. It models the dynamics of the factors but does not rely on a complete characterization of the transitional distribution. We use a partial characterization focussing on the affine form of the first two moments of the transition distribution only. As such we obtain an exact likelihood function in case we would have implemented a Gaussian model. For our non-Gaussian multi-factor CIR model this yields an asymptotically 'correct' approximation.
It can be shown (de Jong (1997) and Duan and Simonato (1998)) that the conditional mean $\mathbf{m}(\mathbf{X}(\mathbf{t}+\boldsymbol{\Delta});\mathbf{X}(\mathbf{t}))$ and conditional variance $\mathbf{Q}(\mathbf{X}(\mathbf{t}+\boldsymbol{\Delta});\mathbf{X}(\mathbf{t}))$ of the factors at time $t+\Delta$ will be affine in the factors. More specifically, given the square root dynamics of the factors in (1) the conditional mean and variance reduce to:

$$\mathbf{m}(\mathbf{X}(\mathbf{t}+\boldsymbol{\Delta});\mathbf{X}(\mathbf{t})) = [m_1(X_1(t+\Delta);X_1(t)),\cdots,m_K(X_K(t+\Delta);X_K(t))]' \tag{17}$$

$$\mathbf{Q}(\mathbf{X}(\mathbf{t}+\boldsymbol{\Delta});\mathbf{X}(\mathbf{t})) = \begin{bmatrix} q_{11}(\cdot) & 0 & \cdots & 0 \\ 0 & q_{22}(\cdot) & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & q_{KK}(\cdot) \end{bmatrix}$$

where each $m_i$and q_{ii}, $i=\{1,...,K\}$, is derived from our CIR model as :

$$m_i(X_i(t+\Delta);X_i(t)) = \theta_i(1-\exp(-\kappa_i\Delta))+\exp(-\kappa_i\Delta)X_i(t) \tag{18}$$

$$q_{ii}(X_i(t+\Delta);X_i(t)) = \frac{\sigma_i^2}{\kappa_i}(\exp(-\kappa_i\Delta)-\exp(-2\kappa_i\Delta))X_i(t)+\theta_i\frac{\sigma_i^2}{2\kappa_i}(1-\exp(-\kappa_i\Delta))^2. \tag{19}$$

Defining $\boldsymbol{\eta}(\mathbf{t})$ as a zero mean unit variance random variable of dimension $(K\times 1)$ the transition equation for $\mathbf{X}(\mathbf{t})$ can be written as:

$$\mathbf{X}(\mathbf{t}+\boldsymbol{\Delta}) = \mathbf{m}(\mathbf{X}(\mathbf{t}+\boldsymbol{\Delta});\mathbf{X}(\mathbf{t}))+\mathbf{Q}(\mathbf{X}(\mathbf{t}+\boldsymbol{\Delta});\mathbf{X}(\mathbf{t}))^{1/2}\cdot\boldsymbol{\eta}(\mathbf{t}). \tag{20}$$

The transition equation intuitively compares to the discrete time equivalent of our state variable processes in equation (1).

3.2 The Kalman Filter

The shape of the term structure at time t is completely determined by the unobserved state variables $\mathbf{X(t)}$. The Kalman filter is a recursive algorithm for computing the mathematical expectation of a hidden state vector $\mathbf{X(t)}$, conditional on observing a history of noisy signals on the hidden state, $E[\mathbf{X(t)}|\mathbf{Y_t}(\boldsymbol{\tau}),\mathbf{Y_{t-1}}(\boldsymbol{\tau}),...,\mathbf{Y_0}(\boldsymbol{\tau})]$. Thus, in order to filter the latent factors we apply the Kalman filter procedure to the state space representation of the model, specified by equations (15) and (20). Denote the predictions about the conditional mean of the state variables at time $t+\Delta$, conditional on the time t values by $\mathbf{X(t+\Delta|t)}$ and those of the conditional variance as $\mathbf{V(t+\Delta|t)}$ (both with obvious dimensions) :

$$\mathbf{X(t+\Delta|t)} = \mathbf{m}\left(\mathbf{X}\left(\mathbf{t}+\mathbf{\Delta}\right);\mathbf{X}\left(\mathbf{t}\right)\right) \tag{21}$$

$$\mathbf{V(t+\Delta|t)} = \mathbf{\Lambda}'\cdot\mathbf{V(t)}\cdot\mathbf{\Lambda}+\mathbf{Q}\left(\mathbf{X}\left(\mathbf{t}+\mathbf{\Delta}\right);\mathbf{X}\left(\mathbf{t}\right)\right), \tag{22}$$

with $\mathbf{\Lambda}$ a diagonal $(K\times K)$ matrix with elements $\Lambda_{ii}=\exp\left(-\kappa_i\Delta\right)$. The performance of these predictions about the drift and variance of the state variables can only be evaluated indirectly through the fit of the yield curve. Given that our model is affine we have that the pricing or fitting errors will be determined as $\boldsymbol{\psi}_{\mathbf{t+\Delta}}(\boldsymbol{\tau};\boldsymbol{\vartheta},\boldsymbol{\phi})=[\psi_{t+\Delta}(\tau_1;\boldsymbol{\vartheta},\boldsymbol{\phi}),...,\psi_{t+\Delta}(\tau_N;\boldsymbol{\vartheta},\boldsymbol{\phi})]'$:

$$\boldsymbol{\psi}_{\mathbf{t+\Delta}}(\boldsymbol{\tau};\boldsymbol{\vartheta},\boldsymbol{\phi})=\mathbf{Y}_{\mathbf{t+\Delta}}(\boldsymbol{\tau})-\mathbf{AA}(\boldsymbol{\tau},\boldsymbol{\vartheta})-\mathbf{BB}(\boldsymbol{\tau},\boldsymbol{\vartheta})\cdot\mathbf{X(t+\Delta|t)}, \tag{23}$$

with a corresponding $(N\times N)$ variance matrix $\mathbf{F}(\mathbf{t}+\mathbf{\Delta};\boldsymbol{\vartheta},\boldsymbol{\phi})$:

$$\mathbf{F}(\mathbf{t}+\mathbf{\Delta};\boldsymbol{\vartheta},\boldsymbol{\phi})=\mathbf{H}(\boldsymbol{\phi})+\mathbf{BB}(\boldsymbol{\tau},\boldsymbol{\vartheta})\cdot\mathbf{V(t+\Delta|t)}\cdot\mathbf{BB}(\boldsymbol{\tau},\boldsymbol{\vartheta})'. \tag{24}$$

The latter consists of two components. The measurement errors of the yield curve with variance matrix $\mathbf{H}$ (see equation (15), $\mathbf{H}$ is a diagonal $(N\times N)$ matrix with parameters h_i^2, $i=\{1,...,N\}$, h_i being the standard deviation of measurement error i, $\boldsymbol{\phi}=[h_1,...,h_N]'$) and the uncertainty introduced by the predictions of the future state variables. Indeed, information about $\mathbf{X(t)}$ stems from two sources: the predicted value, determined by the historic term structures, and the observed vector of interest rates at time t. Both sources contain some error. The prediction because of the innovation in the vector state variables between $t-\Delta$ and t as well as the uncertainty about the estimate, the interest rates because they contain a measurement error. The optimal predictor is formed by combining these two pieces of information, and the Kalman filter solves this problem. The weights given to the interest rates and to the prediction depend on the relative sizes of their covariance matrices, so if the interest rates are very noisy, the updated estimated predictor will differ only little from the ex ante predictor.

Finally, using the observed pricing errors $\boldsymbol{\psi}_{\mathbf{t+\Delta}}(\boldsymbol{\tau};\boldsymbol{\vartheta},\boldsymbol{\phi})$ the predictions are updated resulting in the filtered state variables for time $t+\Delta$ as well as their conditional variance matrix:

$$\begin{aligned}\mathbf{X}(\mathbf{t}+\mathbf{\Delta};\boldsymbol{\vartheta},\boldsymbol{\phi})&=\mathbf{X(t+\Delta|t)}+\mathbf{V(t+\Delta|t)}\cdot\mathbf{BB}(\boldsymbol{\tau};\boldsymbol{\vartheta})'\cdot\left(\mathbf{F}(\mathbf{t}+\mathbf{\Delta};\boldsymbol{\vartheta},\boldsymbol{\phi})\right)^{-\mathbf{1}}\cdot\boldsymbol{\psi}_{\mathbf{t+\Delta}}(\boldsymbol{\tau};\boldsymbol{\vartheta},\boldsymbol{\phi}),\\ \mathbf{V}(\mathbf{t}+\mathbf{\Delta};\boldsymbol{\vartheta},\boldsymbol{\phi})&=\mathbf{V(t+\Delta|t)}-\mathbf{V(t+\Delta|t)}\cdot\mathbf{BB}(\boldsymbol{\tau};\boldsymbol{\vartheta})'\cdot\left(\mathbf{F}(\mathbf{t}+\mathbf{\Delta};\boldsymbol{\vartheta},\boldsymbol{\phi})\right)^{-\mathbf{1}}\cdot\mathbf{BB}(\boldsymbol{\tau};\boldsymbol{\vartheta})\cdot\mathbf{V(t+\Delta|t)},\end{aligned} \tag{25}$$

whereafter the same prediction-updating procedure is performed for the next (noisy) observation, finally yielding the filtered state variables.

Summarizing the above, the Kalman procedure updates the estimation every time a new observation becomes available. The filter first forms an optimal predictor of the unobserved state variables, conditional on the previous estimated values. These estimates for the unobserved state variables are then updated using the information provided by the observed variables. As a by-product the filter provides fitting errors together with their conditional variance that can be used to construct the quasi-(log)likelihood function:

$$\sum_{t=1}^{m} l_t \equiv L_m = -\frac{N \cdot m}{2} \ln(2\pi) - \sum_{t=1}^{m} \frac{1}{2} \left(\ln\left(|\mathbf{F}(\mathbf{t};\boldsymbol{\vartheta},\boldsymbol{\phi})|\right) + \boldsymbol{\psi}'_{\mathbf{t}}(\boldsymbol{\tau};\boldsymbol{\vartheta},\boldsymbol{\phi}) \cdot \left(\mathbf{F}(\mathbf{t};\boldsymbol{\vartheta},\boldsymbol{\phi})\right)^{-1} \cdot \boldsymbol{\psi}_{\mathbf{t}}(\boldsymbol{\tau};\boldsymbol{\vartheta},\boldsymbol{\phi}) \right), \tag{26}$$

which in its turn can be used to find ML estimates of the parameters governing the dynamics of the state variables and the yield curve. Unfortunately, the properties of the ML estimator are unknown for square root factor models. The ML estimator can only be used in case the factors and the measurement errors are normally distributed and is only under these restrictive conditions efficient and asymptotic normal. The exponential affine class of term structure models will in general fail to generate normally distributed factors (the Vasicek model is an exception) so that one has to resort to QML estimators. However, even though the QML estimator has been shown to be consistent and asymptotically normal (Bollerslev and Woolridge (1992)), parameter estimates are likely to be biased within the context of the exponential-affine class with time varying variances. The reason for this inconsistency lies in the fact that the exact factors are not observed and that filtered state variables will enter in the conditional variance matrix $\mathbf{F}(\mathbf{t};\boldsymbol{\vartheta},\boldsymbol{\phi})$ imputing errors in the likelihood function (Duan and Simonato (1998) and de Jong (1997)). While the theoretical properties of the QML estimates based on the Kalman filter remain largely unknown, Monte Carlo simulations show that the bias in the parameter estimates tends to be reasonable small for the relevant parameter combinations of the exponential-affine class of term structure models, even for sample sizes as small as 150 datapoints (Duan and Simonato (1998) and Lund (1997)). In the remainder of the paper, we discard the biases and assume that the QML estimator can be used for statistical inference. That is, following Duan and Simonato (1998), we assume that the estimator of the $((4 * K + N) \times 1)$ parameter vector $\boldsymbol{\zeta}$, $\boldsymbol{\zeta} \equiv [\boldsymbol{\vartheta}', \boldsymbol{\phi}']'$, is consistent and asymptotically normal :

$$\sqrt{m}\left(\hat{\boldsymbol{\zeta}}_m - \boldsymbol{\zeta}_0\right) \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Xi}_{\mathbf{m}}^{-1} \boldsymbol{\Sigma}_{\mathbf{m}} \boldsymbol{\Xi}_{\mathbf{m}}^{-1}\right),$$

where

$$\boldsymbol{\Sigma}_{\mathbf{m}} \equiv \frac{1}{m} \sum_{t=1}^{m} \left(\frac{\partial l_t}{\partial \boldsymbol{\zeta}}\right)' \left(\frac{\partial l_t}{\partial \boldsymbol{\zeta}}\right)$$

and

$$\mathbf{\Xi_m} \equiv \frac{1}{m}\sum_{t=1}^{m}\mathbf{f_t},$$

with $\mathbf{f_t}$ the $((4*K+N)\times(4*K+N))$ matrix

$$\mathbf{f_t} = \frac{\partial E[\mathbf{Y_t}(\boldsymbol{\tau})]'}{\partial\boldsymbol{\zeta}}\mathbf{F}(\mathbf{t};\boldsymbol{\phi})^{-\mathbf{1}}\frac{\partial E[\mathbf{Y_t}(\boldsymbol{\tau})]}{\partial\boldsymbol{\zeta}} + \frac{1}{2}\frac{\partial\mathbf{F}(\mathbf{t};\boldsymbol{\zeta})'}{\partial\boldsymbol{\zeta}}\left[\mathbf{F}(\mathbf{t};\boldsymbol{\zeta})^{-\mathbf{1}}\otimes\mathbf{F}(\mathbf{t};\boldsymbol{\zeta})^{-\mathbf{1}}\right]\left(\frac{\partial\mathbf{F}(\mathbf{t};\boldsymbol{\zeta})}{\partial\boldsymbol{\zeta}}\right).$$

4 Empirical analysis on the LIBOR Yield Curve

4.1 Data description

While data are abundantly available for the USA, such data sets are more difficult to obtain for other countries. The lack of conformable data sets lead us to partially construct the data ourselves as stipulated in Piazzesi (2000, page 17). We reconstructed the yield curves for the different countries from the LIBOR rates and the observations on swap rates. Evidently these samples will not represent the standard type of yield curve because of the presence of financial risk (default risk caused by the fact that we are forced to use non-government rates). Modelling the LIBOR yield curve may however be relevant since most of the interest rate derivatives, e.g. swaps, are priced by use of the financial market interest rates, i.e. the LIBOR discount rate.
Monthly observed LIBOR rates from 1 to 12 month maturities for the USA, Germany and the UK are readily available from Datastream. SWAP rates for maturities of 2 up to 5 years as well. To capture the richness of the term structure while remaining parsimonious, we included the 3 month, 6 month, 12 month and 60 month maturities in our analysis ($N = 4$ maturities). For the 60 month maturity, we need to construct the (zero-coupon) LIBOR yields from the SWAP rates.
First observation is 07/04/1987, last observation is 23/03/1999. This yields 144 monthly observations for each maturity and for each country. The as such constructed (LIBOR) yield dynamics for the selected maturity set is shown in *figures 1a/b/c.* Some descriptive statistics are reported in *table 1.*

Insert *figures 1a/b/c here* ⇎ Insert *table 1* here

We reject the null of no excess kurtosis statistically at the 90% confidence level for the USA at all maturities, for the other two countries even at the 95% level. The null of skewness is statistically rejected at the 95% confidence level for the 3 month, the 6 month and the 1 year maturity for Germany and the UK. Autocorrelations for the 1 month and 12 month lagged yield series are high. Obviously, from *figures 1a/b/c*, there is a remarkable difference in the average term structure and in its dynamics across our sample of countries for the period under consideration. Notice for instance that roughly speaking the USA has a normally inclined term structure for the entire sample, Germany has an inverted one only between '91 and '94, while the UK is inverted in the first halve of the sample. On average, the UK term structure is inverted (see below).

4.2 Estimation results and discussion

In this subsection we present estimation results for the three models under the empirical probability measure P for each of the countries[4]. We estimate one-, two- and three-factor models ($K = 1, 2, 3$ respectively) for each of the LIBOR yield curves. The results for the diffusion parameters and the measurement errors' estimated standard deviations are reported in *tables 2-4* for the USA, Germany and the UK respectively. The dynamics of the state variables in the three-factor model for each of the countries is shown in *figures 2a/b/c*.

Insert *figures 2a/b/c* here ≎ Insert *tables 2-4* here

For all countries we find that the inclusion of extra factors increases the performance of the model in terms of likelihood value as well as in terms of measurement errors' standard deviation sizes. Focussing on the standard deviation sizes, there seems to be some cross-country similarity in what the factors capture. The addition of the second factor basically reduces the size of the measurement errors for the long maturities, while the third factor substantially reduces measurement errors at the short end of the yield curve. The three-factor models price discount bonds well with a typical standard deviation of measurement errors of merely 5 basis points on average. Retaining the assumption of normality of the measurement errors, the model would price bonds correctly roughly up to 20 basis points in 95% of the cases. In practice, however, confidence bounds based on the normality assumption

[4] **Technicalities.** Estimation was performed with the GAUSS® constrained maximum likelihood (CML) routine, using Newton-Raphson's algorithm with step-halving line search. We scaled some parameter values to improve convergence performance. Before initiating the optimization procedure, the parameter space was brute-forcely searched to find suitable starting values for the parameters (a priori grid search). This is done by generating vectors of random drawings within a reasonable range for each parameter (typically 500 drawings). This range compares to the imposed CML bounds ($[0, 3]$ for each κ, $[0, 0.10]$ for each θ, $[0, 2]$ for each σ, $[-\infty, -0.001]$ for each λ, and $[0, 0.05]$ for each h.), but is set tighter for the standard deviations of the measurement errors. Starting values for the latent factors are derived from the unconditional distribution of the factors. That is each factor $X_i(t)$, $i = \{1, ..., K\}$, is started at the instantaneous unconditional mean $X_i(0) = \theta_i$ with instantaneous unconditional variance $\frac{1}{2}\left(\theta_i \sigma_i^2 / \kappa_i\right)$. We approximated the analytical derivative using a two-sided increment approximation, $0.999 \cdot {}^{3*}$ and $1.001 \cdot {}^{3*}$ respectively, where ^{3*} are the optimal parameter values. The convergence criterion, based on the maximum absolute difference in both parameter and functional values between two successive iterations, is set to $1e-4$. The (discrete) sampling frequency considered in the estimation is monthly, i.e. $\Delta = 1/12$. A last point to mention is the handling of negative state variable levels, principally allowed for in the Kalman filter procedure. If this were the case we imposed the factor to be zero, solving the problem with the square root in (1). Thus, the Kalman filter estimator is linear only for $X_i(t) \geq 0$. Prior to estimation on our dataset, we checked our program for correctness by plugging in identical datasets as used by Duan and Simonato (1998, the Fama and Bliss dataset). We were able to reproduce their estimation results precisely (in optimal parameter levels and robust standard errors) raising confidence in the correctness of our program.

may not be valid and filtering (or prediction) errors will have to be considered next to these measurement errors (see below).

In terms of factor dynamics another similarity arises. For the three-factor model we find an extremely inert factor (factor 2) in each country. The estimated halving time for this factor $(\ln(2)/\kappa_2)$ is roughly 385 years for the USA, 86 for the UK and approximately 3500 years for Germany. The second factor is close to being non-stationary. Caution is needed since it is well known in the time series literature that the standard asymptotic properties of the estimators do not hold in the presence of a unit root (since the distribution does not resemble a normal distribution anymore). The other factors have stronger mean reversion and halving times of about 0.3 and 3 years for the USA, 0.33 and 4.5 years for Germany and 0.3 and 1.8 years for the UK. The fast mean-reverting factor is denoted factor 1, the moderate mean-reverting one is factor 3. For an interpretation of the factors see below.

Insert *figure 3* here

The λs within each country are all negative, a necessary condition for positive risk premia. For all three countries, the second factor has by far the highest market price of risk. Risk premia calculated on the basis of equation (13) are quite small for the first three maturities, on average approximately 0.55%, 0.20% and 0.33% respectively for the three countries. The average 5 year maturity risk premium is substantially higher, 0.72% for the USA and the UK, 1.19% for Germany. The latter are plotted in *figure 3* for the three countries.

It is easy to check whether the Feller condition $(2\kappa\theta > \sigma^2)$ holds for our estimates. If not, the factors' upward drifts are not sufficiently large to ensure that the origin is inaccessible. For the three-factor model, all second factors and the third UK factor are prone to become negative sooner or later[5].

Insert *figures 4a/b/c* here

Finally, the estimated factor loadings for the three factor model of each country are plotted in *figures 4a/b/c*. Our reported loading pattern for the three factor model unfortunately does not allow us to neatly isolate the interpretation of our three factors as 'level', 'curvature' and 'slope' of the term structure (as pioneered by Chen and Scott (1993)). Indeed, our third factor is not merely a slope effect, since its downward trend at the long end of the term structure is not negligible, blurring the interpretation especially for the UK. The interpretation of our second factor is not intuitive at all at first sight. Our first factor captures the slope effect unambiguously.

How can these factors be interpreted ? We found correlation coefficients as high as 93% for each of the countries between the third factor and the short term yield, correlation coefficients of on average 80% between the 1year-3month spread and the first factor and correlation

[5] In the technicalities footnote above, we explained how we dealt with negative state variables.

coefficients of over 75% between the 5year-1year spread and the second factor for the US and Germany. The UK second fector is a noteworthy exception, it seems to capture the long term yield with a correlation of 75%.

It is interesting to assess the impact of each single factor in explaining the time series variability of the yields within each market. Together they explain between 95 and 99% of the variation in yields at each end of the term structure. However, the bulk is accounted for by factor 3 alone, around 85% for each country and at each end of the term structure. Factor 2 essentially makes up the difference while factor 1 is only marginal in adding explanatory power to their combination and on its own. However, from the RMSE and the standard deviation sizes of the measurement errors, the usefulness of the inclusion of a first factor was clear. Factor 2 and especially factor 1 capture the cross-sectional dimension of the term structure.

4.3 Diagnostic checks

Diagnostic checking is a necessary requirement in order to make assessments about model misspecifications. There is a battery of methods put forward in the literature to assess the quality of the derived model. A formal way to test the model statistically is to perform a robust Lagrange multiplier test, as in Duan and Simonato (1998). This test essentially opposes the cross-sectional restrictions imposed by the model against the time series dimension restrictions[6]. From the p-values from the χ^2-test in *tables 2-4*, we find a firm statistical rejection of the model, a usual finding in the literature.

Insert *figures 5a/b/c* $\Leftrightarrow$ Insert *tables 5* and *6* here

Table 5 reports the RMSE statistics for the different models, revealing that the inclusion of a second and third factor dramatically improves the fit of the term structure (which was already suggested by the firm increase in quasi-loglikelihood value of *tables 2-4*), especially at the long end of the term structure. The RMSE statistic takes the errors implied by the filtering methodology into account next to the above mentioned measurement errors, making them more suitable for model evaluation. From the table, one can see that the first factor fails to fit the long end of the term structure relative to the short end, with errors more than twice as high for the long end. The inclusion of the second factor tackles this pricing problem to a large extent at the cost of somewhat higher RMSE at the other maturities. The inclusion of the third factor corrects for this and shaves off RMSE over the entire maturity spectrum. To complement these RMSE statistics, we included pricing error histograms (*figures 5a/b/c*) and the corresponding *table 6*. From these one has to remark that the three-factor model

[6] This test is comprable but more general than the test as proposed in de Jong (1997) since more flexibility in the alternative specification is allowed for.

performs well, and this in spite of the statistical model rejection[7]. For example, the probability of mispricing more than 1% is less than 5% in all cases, the chance of making an error of less than 10 basis points goes from 20% to 50% depending on the country and maturity considered. Focussing on the histograms, we see graphically that roughly most of the probability mass is situated between errors of -30 and $+30$ basis points. Notice the presence of an outlier in each histogram. This is due to misspecification. Indeed, at the initialisation of the filter we impose it to start at its unconditional mean and variance, while this may be way off the mark. Some researchers chose to omit the likelihood of the first couple of observations to reduce the impact of this likely misspecification. Obviously, we could have tabulated the means and autocorrelations of the pricing errors, rendering more support to the above graphical statements, namely that the pricing errors are biased, autocorrelated, and by far non-normally distributed. All this is strong evidence against the multi-factor CIR model. The reasons for this apparent failure are to be sought in the restrictive and rigid constraints imposed by the model (non-negativity is one of them). The multi-factor model seems to be performant in mimicking the shape of the term structure, but when it comes to mimicking the dynamics of the observed term structure, the model breaks down.

Another (less formal) way to assess the quality of the model is to examine the fit of the modelled term structure to the actual term structure visually. There is an abundance of possible time points at which to confront the model versus reality, raising the question which datapoints to pick out. We decided to plot three specific term structures for each country (together with their $K = \{1, 2, 3\}$ model predictions); (i) the average yield curve (*figures 6a/b/c*), (ii) the 'steepest normal'[8] yield curve (*figures 7a/b/c*) and (iii) the 'steepest inverted' average yield curve (*figures 8a/b/c*). Note that the latter two series are extreme observations, making it interesting to see how the model behaves in these circumstances.

Insert *figures 6a/b/c – 7a/b/c – 8a/b/c* here

From these term structure graphs, one notices that the multi-factor CIR models are rather flexible in the shapes that it may assume. We notice the superiority of the three-factor model over the one- and two-factor models in fitting the average term structure. For the two other series of graphs, the fit is reasonable in general and the superiority of the three-factor model is frequently clear.

Insert *figures 9a/b/c* here

[7] In spite of the statistical rejection and in order to demonstrate the usefulness of the model, we plan to oppose our one period ahead term structure prediction against a random walk model where the one period ahead forecast of the term structure is the current term structure.

[8] 'Steepest normal' is to be understood as there where the spread between the 5 year rate and the 3 month rate is maximally positive, 'steepest inverted' as there where the same spread is maximally negative.

Another test consists in simply regressing yields on factors and comparing the regression coefficients with the factor loadings. If the model is any good, these should be comparable in shape and magnitude. We only included the test for the three-factor model. From *figures 8a/b/c*, we can graphically see that the model performs well in this respect, though there is no econometric method available to formalize this statement.

4.4 Explaining Correlations

Multi-factor models are designed to model the correlations among the different bonds by a limited number of factors. If any good, these models should also account for the correlation between markets. This will be investigated by means of a correlation structure analysis. *Table 7* provides the correlation matrix of a selected number of bond yields for the USA, Germany and the UK. Interest rates taken are the 3-month, 6-month, 1 year and 5 year rates.

Insert *table 7* here

It is clear from the tabulated correlations that single-factor models must fail dramatically. Typically, the within market correlations tend to be the highest, of the order of 85% or more. Also, we find that the correlations tend to decline with the difference in maturities of the bond considered. This pattern is observed in all three countries and calls for a multi-factor approach to the term structure. Although within market correlations are without exception the highest, correlations between markets are not negligible. Relatively high correlations between the USA and the UK as well as between Germany and the UK are observed while correlations between Germany and the USA are weak and only exist for the longer maturities at best. If multi-factor models are to be used for pricing or risk management, these models should also be able to model the observed international correlations. The one-factor models will fail to fit simultaneously the high correlations between the USA and the UK and that between the UK and Germany on the one hand and the weak correlations between the USA and Germany on the other hand. Here we focus on the three-factor models in trying to replicate the observed correlations.

The within and between market (unconditional) correlations can be recovered from the estimated parameters of the model and the covariance matrices of the factors. More specifically, considering bonds with maturities τ_1 and τ_2 for markets H and F, respectively, we have that :

$$\operatorname{corr}\left(Y_H(\tau_1), Y_F(\tau_2)\right) = \frac{\sum_{i=1}^{K}\sum_{j=1}^{K} B_{i,H}\left(\tau_1\right) B_{j,F}\left(\tau_2\right) cov\left(X_{i,H}, X_{j,F}\right)}{\sqrt{var\left(Y_H(\tau_1)\right)}\sqrt{var\left(Y_F(\tau_2)\right)}}. \tag{27}$$

Note that we have imposed zero correlation for all maturities and countries with the measurement error. Obviously, this condition might breakdown in practice. We did not impose the conditions that the factors driving a market's yield curve should be uncorrelated. Factors can be correlated between markets. Note that the correlation can be decomposed into K^2 components, each of these components focuses on a specific relation among the factors. We are therefore able to determine the contribution of the co-movement among factors (internationally) to the final correlation of the yields. This decomposition together with the computed values for the model's correlations are tabulated in tables 8-11.

Insert *tables 8-11* here

First, the models' implied correlations are broadly speaking in line with the observed within market correlations. In this respect the model seems to match and model the data rather well. The deviations between the implied correlation and the observed one is due to covariation with the measurement errors. Note, however, that the assumption of uncorrelated factors does not verify completely. In table 8 there is evidence that the alternative factors have some correlation, most outspoken in the USA. Thus the model tends to reproduce the observed within market correlations rather well.

Tables 9-11 present the implied correlation of yield curves between countries as well as the contribution of each combination of the factors. Again, we see that the empirical correlations tend to be reproduced rather well by the model. The interesting aspect of these tables is that they give us a clear view of what causes the correlations. One could ague for instance that the German and USA market are not correlated, especially so at the short end of their term structures. However, there is a clear correlation between the German third factor and the UK third factor and between the USA third factor and the UK third factor. Hence, a tentative explanation would be to assume that there are two autonomous blocks in the world, Germany and the USA, being linked only at the long end of their term structures, while the UK is somewhere in between, undergoing the effects of both blocks. Whatever the interpretation, an important finding is that the factors for each of the countries tend to have strong correlations internationally, suggesting that some of these factors may be international factors.

5 Conclusion

In this paper we have estimated a multi-factor CIR model for the yield curves of the USA, Germany and the UK for the period 1987-1999. To identify the latent factors we applied the Kalman filter procedure, which efficiently incorporates both the cross-sectional information in the yield curve as well as the intertemporal information for the interest rates.

The high correlation among the yields within a market has been the main motivation for explaining yield curve dynamics by means of a limited set of factors. We find that the three-factor version of the CIR model broadly reproduces the high correlations observed within each of the markets. Moreover, the CIR model (estimated separately for each of the countries) reproduces the observed correlations between markets as well. An important finding here is that the factors for each of the countries tend to have strong correlations internationally, suggesting that some of these factors may be common factors.
We found that the three-factor models give an economically more or less adequate description of the yield curves. This conclusion is in line with previous research, indicating that two to three factors are necessary but sufficient for describing the yield curve dynamics by means of multi-factor CIR models. In line as well with the literature, we find firm evidence to statistically reject the CIR multi-factor model as an adequate method of describing the term structure. Pricing errors are biased, autocorrelated, and non-normal. We argue that the multi-factor model performs well in capturing the dynamics of the term structure, but fails to mimick the shape of the term structure simultaneously. An interesting and promising way to tackle this problem is to allow for a more flexible parametrization of the market price of risk, as in Duarte (1999).
A logical next step is then to try to model the correlations both within and between markets by means of a limited set of national and common factors. Such extension may prove essential for applications of the CIR-models in international finance. We plan to pursue this direction of research in the near future.

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Table 1: **Summary statistics of the USA, Germany and UK yield data.**

USA

Maturity	3	6	12	60
Mean	6.04	6.09	6.20	7.28
Stan.Dev.	1.76	1.72	1.65	1.46
Excess kurtosis	-0.6660	-0.6629	-0.7101	-1.3009
	$(.10)^a$	$(.10)^a$	$(.08)^a$	$(.00)^a$
Skewness	0.2572	0.2527	0.2288	0.1968
	$(.20)^b$	$(.21)^b$	$(.26)^b$	$(.33)^b$
$\rho(1)$	0.9973	0.9971	0.9968	0.9967
$\rho(12)$	0.9973	0.9971	0.9968	0.9967

GERMANY

Maturity	3	6	12	60
Mean	5.78	5.77	5.71	6.47
Stan.Dev.	2.32	2.27	2.15	1.48
Excess kurtosis	-1.4059	-1.3815	-1.3166	-0.9555
	$(.00)^a$	$(.00)^a$	$(.00)^a$	$(.02)^a$
Skewness	0.4038	0.4116	0.4434	0.2654
	$(.05)^b$	$(.04)^b$	$(.03)^b$	$(.19)^b$
$\rho(1)$	0.9987	0.9986	0.9984	0.9979
$\rho(12)$	0.9987	0.9986	0.9984	0.9979

UK

Maturity	3	6	12	60
Mean	8.91	8.81	8.65	8.79
Stan.Dev.	3.09	2.95	2.69	1.96
Excess kurtosis	-0.9455	-0.9037	-0.8654	-0.8008
	$(0.02)^a$	$(.03)^a$	$(.03)^a$	$(.05)^a$
Skewness	0.6548	0.6534	0.6134	0.2902
	$(.00)^b$	$(.00)^b$	$(.00)^b$	$(.15)^b$
$\rho(1)$	0.9964	0.9963	0.9964	0.9965
$\rho(12)$	0.9964	0.9963	0.9964	0.9965

Mean and standard deviations reported in percentages p.a. Maturity expressed in months. $\rho(k)$ is the kth order autocorrelation. Superscript ^{a} denotes the significance level in percentage at which the null of no excess kurtosis can be rejected. Superscript ^{b} denotes the significance level in percentage at which the null of skewness can be rejected.

Table 2: **Estimated parameters for K-factor models for the USA.**

	K = 1	K = 2	K = 3
κ_1	0.2575	0.4185	2.4274
	(0.0538)	(0.0885)	(0.1698)
θ_1	0.0568	0.0372	0.0196
	(0.0096)	(0.0077)	(0.0005)
σ_1	0.0463	0.0550	0.1071
	(0.0032)	(0.0099)	(0.0603)
λ_1	-0.1180	-0.0125	-0.0090
	(0.0459)	(0.0849)	(0.0450)
κ_2		0.0002	0.0018
		(0.1082)	(0.0207)
θ_2		0.0001	0.0001
		(0.0523)	(0.0012)
σ_2		0.0503	0.0435
		(0.0204)	(0.0233)
λ_2		-0.1855	-0.2993
		(0.1041)	(0.0171)
κ_3			0.2285
			(0.1018)
θ_3			0.0292
			(0.0134)
σ_3			0.0865
			(0.0317)
λ_3			-0.1054
			(0.1084)
h_{3m}	0.0014	0.0012	0.0006
	(0.0002)	(0.0001)	(0.0001)
h_{6m}	0.0005	0.0000	0.0000
	(0.2296)	(0.0008)	(0.0007)
h_{12m}	0.0019	0.0014	0.0006
	(0.0001)	(0.0001)	(0.0001)
h_{60m}	0.0073	0.0008	0.0005
	(0.0003)	(0.0009)	(0.0008)
Log L	2551	2772	2890
χ^2	10 276 661	104 135	2 437 379
$p-value$	0.00000	0.00000	0.00000

Table 3: **Estimated parameters for K-factor models for Germany.**

	K = 1	K = 2	K = 3
κ_1	0.3350	0.3368	1.8788
	(0.0492)	(0.0493)	(0.1233)
θ_1	0.0490	0.0209	0.0234
	(0.0064)	(0.0030)	(0.0015)
σ_1	0.0415	0.0538	0.0918
	(0.0035)	(0.0063)	(0.0419)
λ_1	-0.1056	-0.0010	-0.0214
	(0.0469)	(0.0490)	(0.1004)
κ_2		0.0000	0.0002
		(0.0899)	(0.0187)
θ_2		0.0000	0.0000
		(0.0324)	(0.0011)
σ_2		0.0336	0.0139
		(0.0117)	(0.0081)
λ_2		-0.2827	-0.6788
		(0.0888)	(0.0199)
κ_3			0.1555
			(0.0129)
θ_3			0.0440
			(0.0015)
σ_3			0.0761
			(0.0249)
λ_3			-0.0010
			(0.0118)
h_{3m}	0.0017	0.0015	0.0008
	(0.0002)	(0.0001)	(0.0001)
h_{6m}	0.0000	0.0000	0.0001
	(4.0801)	(0.0002)	(0.0006)
h_{12m}	0.0025	0.0017	0.0003
	(0.0002)	(0.0001)	(0.0001)
h_{60m}	0.0068	0.0005	0.0000
	(0.0003)	(0.0023)	(0.0000)
Log L	2523	2755	2966
χ^2	7 006 068	57 670	171 569
$p - value$	0.00000	0.00000	0.00000

Table 4: **Estimated parameters for K-factor models for the UK.**

	K = 1	K = 2	K = 3
κ_1	0.3346	0.6505	2.3033
	(0.0479)	(0.0517)	(0.1537)
θ_1	0.0802	0.0395	0.0365
	(0.0096)	(0.0027)	(0.0006)
σ_1	0.0550	0.0793	0.1149
	(0.0058)	(0.0138)	(0.0710)
λ_1	-0.0110	-0.0010	-0.0010
	(0.0418)	(0.0436)	(0.0070)
κ_2		0.0001	0.0008
		(0.0521)	(0.0070)
θ_2		0.0001	0.0001
		(0.0357)	(0.0008)
σ_2		0.0517	0.0483
		(0.0183)	(0.0102)
λ_2		-0.1523	-0.3068
		(0.0531)	(0.0047)
κ_3			0.3756
			(0.0078)
θ_3			0.0180
			(0.0012)
σ_3			0.1274
			(0.0527)
λ_3			-0.0010
			(0.0065)
h_{3m}	0.0022	0.0019	0.0009
	(0.0003)	(0.0001)	(0.0001)
h_{6m}	0.0000	0.0000	0.0000
	(0.0910)	(0.0004)	(0.0000)
h_{12m}	0.0027	0.0018	0.0006
	(0.0002)	(0.0001)	(0.0001)
h_{60m}	0.0086	0.0012	0.0009
	(0.0004)	(0.0017)	(0.0006)
Log L	2365	2574	2725
χ^2	87 257	38 968	83 279
$p-value$	0.00000	0.00000	0.00000

Table 5: RMSE for K factor models.

RMSE		K=1	K=2	K=3
3 month	USA	29.7	36.8	31.2
	GER	27.7	28.9	25.9
	UK	45.3	65.3	56.1
6 month	USA	32.2	39.6	33.9
	GER	28.3	30.9	27.6
	UK	46.8	65.8	56.2
1 year	USA	41.1	45.3	37.7
	GER	39.7	37.6	28.3
	UK	56.0	66.9	54.2
5 year	USA	78.7	48.9	39.7
	GER	71.7	37.9	33.6
	UK	93.1	59.2	50.1

RMSE is reported in basis points.

Table 6: **Fitting the yield curve: fitting errors.**

		$P[\|\psi\| < 0.01\%]$	$P[\|\psi\| < 0.1\%]$	$P[\|\psi\| < 0.5\%]$	$P[\|\psi\| < 1.0\%]$	$P[\|\psi\| < 3.0\%]$
3 month	USA	4.9	34.7	91.0	99.3	100.0
	GER	6.3	47.2	93.7	99.3	100.0
	UK	2.8	33.3	86.1	95.1	99.3
6 month	USA	2.8	29.9	88.2	99.3	100.0
	GER	5.6	42.4	93.7	98.6	100.0
	UK	4.9	30.6	83.3	95.1	99.3
1 year	USA	1.4	24.3	84.7	98.6	100.0
	GER	2.8	41.0	93.7	98.6	100.0
	UK	2.1	28.5	80.6	95.1	99.3
5 year	USA	1.4	21.5	84.0	97.9	100.0
	GER	4.2	25.0	93.7	98.6	100.0
	UK	2.1	21.5	84.7	96.5	99.3

The probabilities in the table are reported as percentages.

Table 7: **Correlations within and between bond markets.**

	$\mathbf{r}_{3m}^{USA}$	$\mathbf{r}_{6m}^{USA}$	$\mathbf{r}_{1y}^{USA}$	$\mathbf{r}_{5y}^{USA}$	$\mathbf{r}_{3m}^{GER}$	$\mathbf{r}_{6m}^{GER}$	$\mathbf{r}_{1y}^{GER}$	$\mathbf{r}_{5y}^{GER}$	$\mathbf{r}_{3m}^{UK}$	$\mathbf{r}_{6m}^{UK}$	$\mathbf{r}_{1y}^{UK}$	$\mathbf{r}_{5y}^{UK}$
$\mathbf{r}_{3m}^{USA}$	·											
$\mathbf{r}_{6m}^{USA}$	.99	·										
$\mathbf{r}_{1y}^{USA}$	.99	.99	·									
$\mathbf{r}_{5y}^{USA}$	.85	.87	.90	·								
$\mathbf{r}_{3m}^{GER}$	-.05	-.06	-.05	.21	·							
$\mathbf{r}_{6m}^{GER}$	.00	.00	.00	.26	.99	·						
$\mathbf{r}_{1y}^{GER}$	.07	.06	.07	.32	.99	.99	·					
$\mathbf{r}_{5y}^{GER}$	.20	.20	.22	.46	.88	.90	.93	·				
$\mathbf{r}_{3m}^{UK}$	.72	.70	.68	.74	.59	.63	.67	.66	·			
$\mathbf{r}_{6m}^{UK}$	.74	.72	.71	.76	.57	.61	.66	.66	.99	·		
$\mathbf{r}_{1y}^{UK}$	.76	.75	.74	.79	.55	.59	.64	.67	.99	.99	·	
$\mathbf{r}_{5y}^{UK}$	.70	.71	.72	.86	.58	.62	.67	.78	.90	.91	.94	·

Bold numbers refer to correlations across bond rates within the same market (country). Normal font is used for correlations of bonds between markets (countries). The correlations were computed on a sample containing monthly observation of the yield curves.

Table 8: **Loading factor weighted correlations within the same market .**

	(3,3)	(3,2)	(3,1)	(2,3)	(2,2)	(2,1)	(1,3)	(1,2)	(1,1)	total	real
	Correlation between US bond rates										
corr(y_{3m}^{usa}, y_{6m}^{usa})	1.00	-.06	-.01	-.06	.07	-.01	-.01	-.01	.05	.95	.99
corr(y_{3m}^{usa}, y_{1y}^{usa})	.99	-.07	-.01	-.06	.08	-.00	-.01	-.01	.03	.95	.99
corr(y_{3m}^{usa}, y_{5y}^{usa})	.88	-.14	-.00	-.05	.17	-.00	-.01	-.02	.01	.82	.85
corr(y_{6m}^{usa}, y_{1y}^{usa})	.99	-.07	-.01	-.06	.08	-.01	-.01	-.01	.03	.96	.99
corr(y_{6m}^{usa}, y_{5y}^{usa})	.88	-.15	-.00	-.05	.19	-.00	-.01	-.02	.01	.84	.87
corr(y_{1y}^{usa}, y_{5y}^{usa})	.89	-.15	-.00	-.06	.21	-.00	-.01	-.01	.00	.88	.90
	Correlation between German bond rates										
corr(y_{3m}^{ger}, y_{6m}^{ger})	.99	-.01	-.04	-.01	.00	-.00	-.05	-.00	.09	.97	.99
corr(y_{3m}^{ger}, y_{1y}^{ger})	1.01	-.02	-.03	-.01	.00	-.00	-.05	-.00	.07	.97	.99
corr(y_{3m}^{ger}, y_{5y}^{ger})	1.09	-.14	-.01	-.01	.01	-.00	-.05	-.03	.02	.89	.88
corr(y_{6m}^{ger}, y_{1y}^{ger})	1.01	-.02	-.03	-.01	.00	-.00	-.04	-.00	.06	.97	.99
corr(y_{6m}^{ger}, y_{5y}^{ger})	1.09	-.14	-.01	-.01	.01	-.00	-.04	-.02	.02	.90	.90
corr(y_{1y}^{ger}, y_{5y}^{ger})	1.11	-.14	-.01	-.02	.02	-.00	-.03	-.02	.01	.92	.93
	Correlation between UK bond rates										
corr(y_{3m}^{uk}, y_{6m}^{uk})	.75	.05	-.01	.05	.03	.00	-.01	.00	.06	.93	.99
corr(y_{3m}^{uk}, y_{1y}^{uk})	.75	.06	-.00	.05	.03	.00	-.01	.00	.04	.93	.99
corr(y_{3m}^{uk}, y_{5y}^{uk})	.55	.17	-.00	.03	.09	.00	-.00	.01	.01	.85	.90
corr(y_{6m}^{uk}, y_{1y}^{uk})	.75	.06	-.00	.05	.03	.00	-.01	.00	.03	.93	.99
corr(y_{6m}^{uk}, y_{5y}^{uk})	.55	.17	-.00	.04	.09	.00	-.00	.01	.01	.86	.91
corr(y_{1y}^{uk}, y_{5y}^{uk})	.55	.17	-.00	.04	.11	.00	-.00	.01	.01	.88	.94

Note that correlation contributions (i, j) and (j, i), $j \neq i$ should not be identical since the respective loadings are different, certainly at the long end of the term structure.

Table 9: **Correlations between USA and German markets and state factor correlations.**

	(3,3)	(3,2)	(3,1)	(2,3)	(2,2)	(2,1)	(1,3)	(1,2)	(1,1)	total	real
corr$(y_{3m}^{usa}, y_{3m}^{ger})$	.02	.00	-.25	.16	-.00	.01	.03	-.00	.01	-.01	-.05
corr$(y_{3m}^{usa}, y_{6m}^{ger})$	.02	.01	-.21	.16	-.00	.01	.03	-.00	.01	.02	.00
corr$(y_{3m}^{usa}, y_{1y}^{ger})$	.02	.01	-.15	.16	-.00	.01	.03	-.00	.01	.08	.07
corr$(y_{3m}^{usa}, y_{5y}^{ger})$	.02	.06	-.05	.17	-.00	.00	.03	-.03	.00	.20	.20
corr$(y_{6m}^{usa}, y_{3m}^{ger})$	.02	.00	-.25	.17	-.00	.01	.02	-.00	.01	-.01	-.06
corr$(y_{6m}^{usa}, y_{6m}^{ger})$	.02	.01	-.21	.17	-.00	.01	.02	-.00	.01	.03	.00
corr$(y_{6m}^{usa}, y_{1y}^{ger})$	.02	.01	-.15	.17	-.00	.01	.02	-.00	.01	.08	.06
corr$(y_{6m}^{usa}, y_{5y}^{ger})$	.02	.06	-.05	.18	-.00	.00	.03	-.03	.00	.21	.20
corr$(y_{1y}^{usa}, y_{3m}^{ger})$	.02	.00	-.26	.19	-.00	.02	.02	-.00	.01	-.00	-.05
corr$(y_{1y}^{usa}, y_{6m}^{ger})$	.02	.01	-.21	.19	-.00	.01	.02	-.00	.01	.04	.00
corr$(y_{1y}^{usa}, y_{1y}^{ger})$	.02	.01	-.16	.19	-.00	.01	.02	-.00	.00	.09	.07
corr$(y_{1y}^{usa}, y_{5y}^{ger})$	.02	.06	-.05	.20	-.00	.00	.02	-.02	.00	.23	.22
corr$(y_{5y}^{usa}, y_{3m}^{ger})$	.02	.00	-.22	.41	-.00	.04	.00	-.00	.00	.25	.21
corr$(y_{5y}^{usa}, y_{6m}^{ger})$	.02	.00	-.19	.41	-.00	.03	.00	-.00	.00	.28	.26
corr$(y_{5y}^{usa}, y_{1y}^{ger})$	.02	.01	-.14	.42	-.00	.02	.00	-.00	.00	.33	.32
corr$(y_{5y}^{usa}, y_{5y}^{ger})$	.02	.05	-.05	.45	-.01	.01	.00	-.00	.00	.47	.46

Table 10: **Correlations between USA and UK markets and state factor correlations.**

	(3,3)	(3,2)	(3,1)	(2,3)	(2,2)	(2,1)	(1,3)	(1,2)	(1,1)	total	real
corr$(y_{3m}^{usa}, y_{3m}^{uk})$	.55	.03	-.09	.06	.03	.02	.06	-.01	.04	.68	.72
corr$(y_{3m}^{usa}, y_{6m}^{uk})$	.55	.04	-.08	.06	.03	.01	.06	-.01	.03	.69	.74
corr$(y_{3m}^{usa}, y_{1y}^{uk})$	.55	.04	-.06	.06	.04	.01	.06	-.01	.02	.71	.76
corr$(y_{3m}^{usa}, y_{5y}^{uk})$	.40	.12	-.02	.04	.10	.00	.04	-.03	.01	.67	.70
corr$(y_{6m}^{usa}, y_{3m}^{uk})$	.55	.03	-.10	.06	.03	.02	.05	-.01	.03	.67	.70
corr$(y_{6m}^{usa}, y_{6m}^{uk})$	.55	.04	-.08	.06	.03	.02	.05	-.01	.02	.69	.72
corr$(y_{6m}^{usa}, y_{1y}^{uk})$	.55	.04	-.06	.06	.04	.01	.05	-.01	.02	.71	.75
corr$(y_{6m}^{usa}, y_{5y}^{uk})$	.40	.12	-.02	.04	.11	.00	.04	-.02	.01	.67	.71
corr$(y_{1y}^{usa}, y_{3m}^{uk})$	.56	.03	-.10	.07	.03	.02	.03	-.00	.02	.67	.68
corr$(y_{1y}^{usa}, y_{6m}^{uk})$	.56	.04	-.08	.07	.04	.02	.03	-.00	.02	.68	.71
corr$(y_{1y}^{usa}, y_{1y}^{uk})$	.56	.04	-.06	.07	.05	.01	.03	-.01	.01	.71	.74
corr$(y_{1y}^{usa}, y_{5y}^{uk})$	.41	.12	-.02	.05	.12	.00	.02	-.02	.00	.69	.72
corr$(y_{5y}^{usa}, y_{3m}^{uk})$	.49	.03	-.08	.15	.08	.05	.01	-.00	.00	.72	.74
corr$(y_{5y}^{usa}, y_{6m}^{uk})$	.49	.03	-.07	.15	.08	.04	.01	-.00	.00	.73	.76
corr$(y_{5y}^{usa}, y_{1y}^{uk})$	.49	.04	-.05	.15	.10	.03	.01	-.00	.00	.76	.79
corr$(y_{5y}^{usa}, y_{5y}^{uk})$	.36	.10	-.02	.11	.27	.01	.01	-.00	.00	.83	.86

Table 11: **Correlations between UK and German markets and state factor correlations.**

	(3,3)	(3,2)	(3,1)	(2,3)	(2,2)	(2,1)	(1,3)	(1,2)	(1,1)	total	real
$\mathrm{corr}(y^{ger}_{3m}, y^{uk}_{3m})$	.55	.09	.08	-.01	.00	-.00	-.17	-.01	.05	.58	.59
$\mathrm{corr}(y^{ger}_{3m}, y^{uk}_{6m})$	.55	.10	.07	-.01	.00	-.00	-.17	-.01	.04	.57	.57
$\mathrm{corr}(y^{ger}_{3m}, y^{uk}_{1y})$	.55	.12	.05	-.01	.00	-.00	-.17	-.01	.03	.55	.55
$\mathrm{corr}(y^{ger}_{3m}, y^{uk}_{5y})$	.40	.32	.01	-.01	.01	-.00	-.12	-.03	.01	.59	.58
$\mathrm{corr}(y^{ger}_{6m}, y^{uk}_{3m})$	.55	.09	.08	-.01	.00	-.00	-.14	-.01	.04	.60	.63
$\mathrm{corr}(y^{ger}_{6m}, y^{uk}_{6m})$	.55	.10	.07	-.01	.00	-.00	-.14	-.01	.03	.59	.61
$\mathrm{corr}(y^{ger}_{6m}, y^{uk}_{3y})$	.55	.12	.05	-.01	.00	-.00	-.14	-.01	.02	.58	.59
$\mathrm{corr}(y^{ger}_{6m}, y^{uk}_{5y})$	.40	.32	.02	-.01	.01	-.00	-.10	-.03	.01	.62	.62
$\mathrm{corr}(y^{ger}_{1y}, y^{uk}_{3m})$	.56	.09	.08	-.01	.00	-.00	-.10	-.01	.03	.64	.67
$\mathrm{corr}(y^{ger}_{1y}, y^{uk}_{6m})$	.56	.10	.07	-.01	.00	-.00	-.10	-.01	.03	.63	.66
$\mathrm{corr}(y^{ger}_{1y}, y^{uk}_{1y})$	.56	.12	.05	-.01	.00	-.00	-.11	-.01	.02	.62	.64
$\mathrm{corr}(y^{ger}_{1y}, y^{uk}_{5y})$	.41	.33	.02	-.01	.01	-.00	-.08	-.02	.01	.66	.67
$\mathrm{corr}(y^{ger}_{5y}, y^{uk}_{3m})$	.60	.10	.09	-.11	.02	-.04	-.04	-.00	.01	.64	.66
$\mathrm{corr}(y^{ger}_{5y}, y^{uk}_{6m})$	.60	.11	.07	-.11	.03	-.03	-.04	-.00	.01	.64	.66
$\mathrm{corr}(y^{ger}_{5y}, y^{uk}_{1y})$	.60	.13	.05	-.11	.03	-.02	-.04	-.00	.01	.65	.67
$\mathrm{corr}(y^{ger}_{5y}, y^{uk}_{5y})$	.44	.35	.02	-.08	.08	-.01	-.03	-.01	.00	.77	.78

Figure 9a : USA 3 factor model : model test.

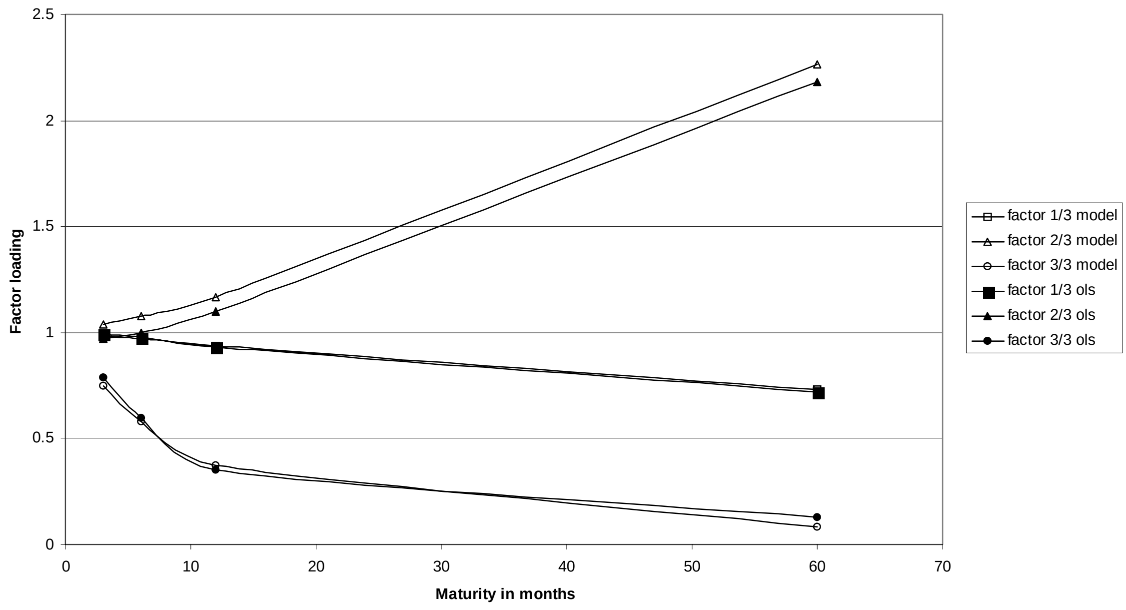

Figure 9b : Germany 3 factor model : model test.

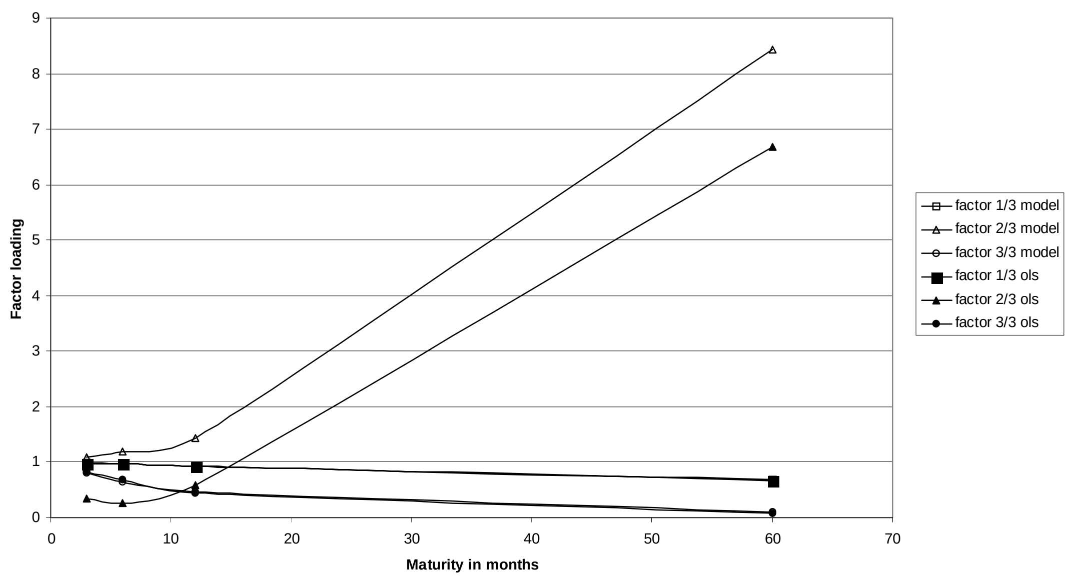

Figure 9c : UK 3 factor model : model test.

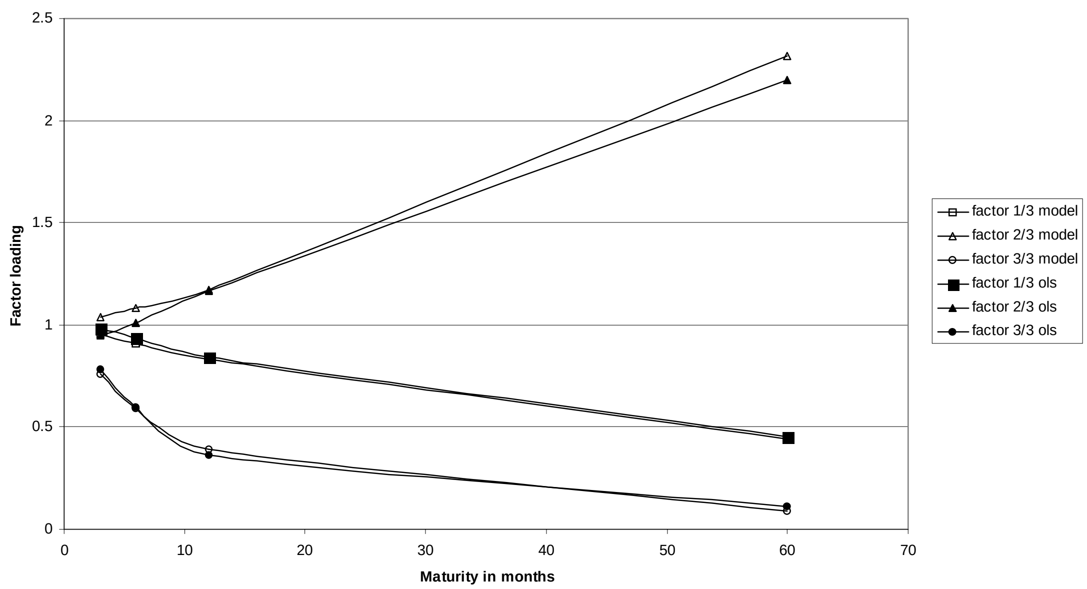

Figure 5a : USA 3 factor model : histogram of fitting errors.

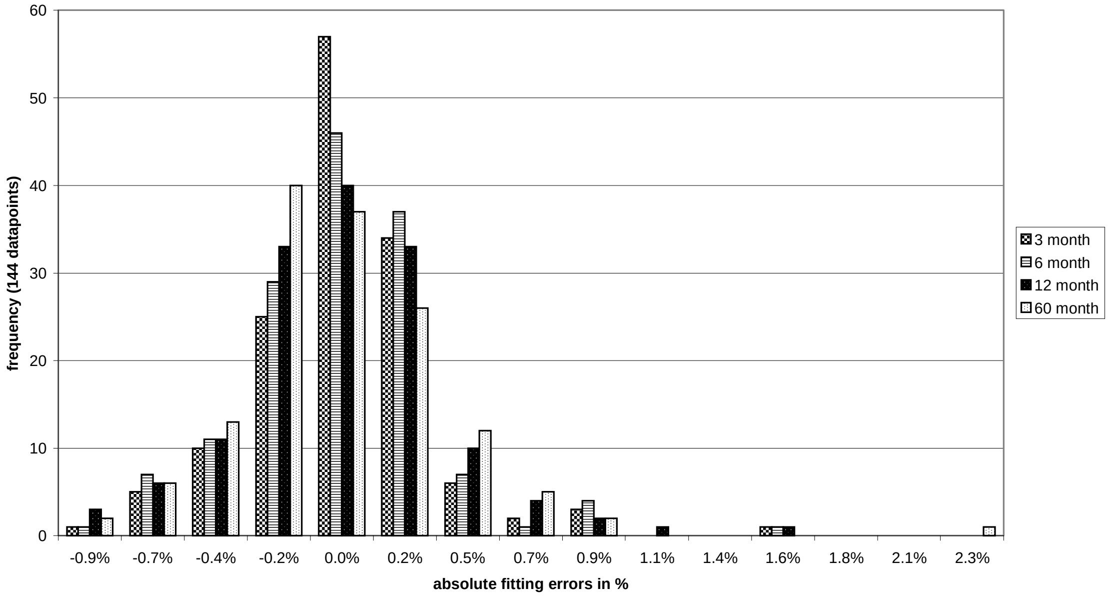

Figure 5b : Germany 3 factor model : histogram of fitting errors.

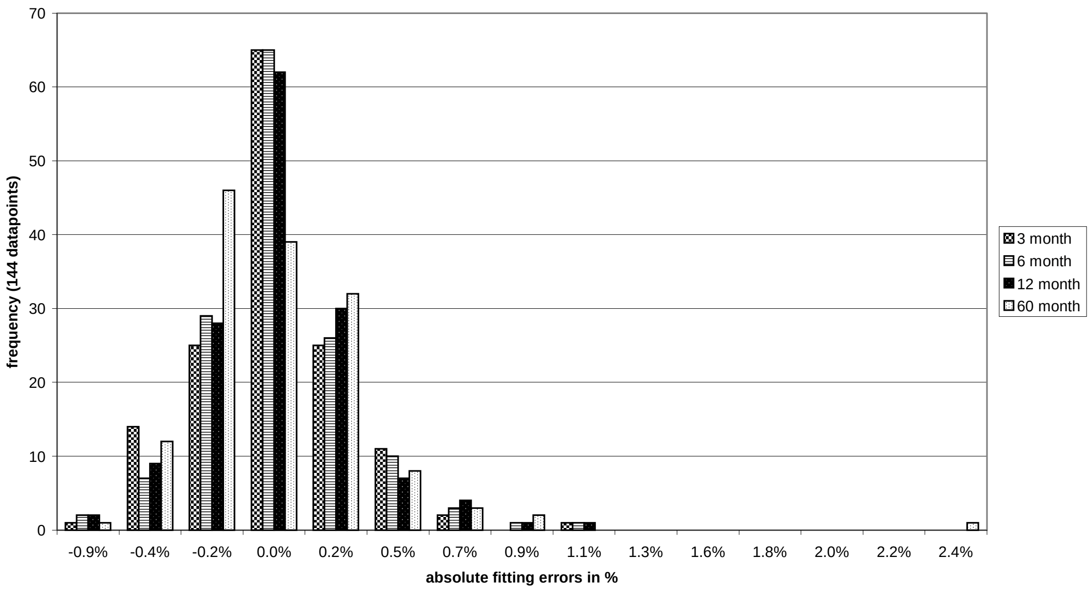

Figure 5c : UK 3 factor model : histogram of fitting errors.

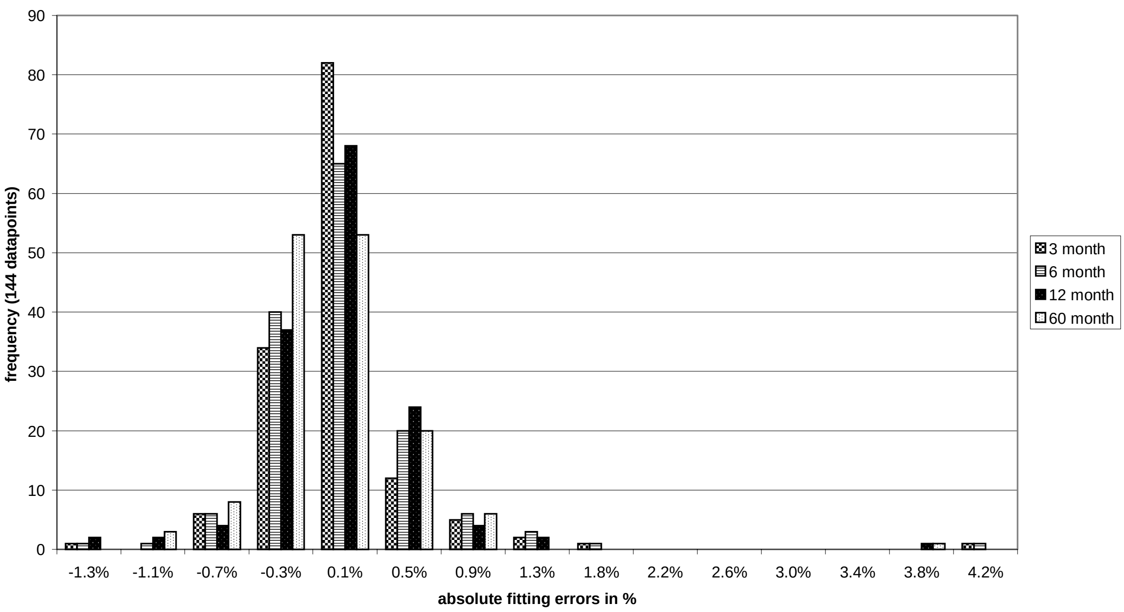

Figure 8a : USA : steepest inverted term structure (month 28, yield difference -0.00588) and the K factor model fits.

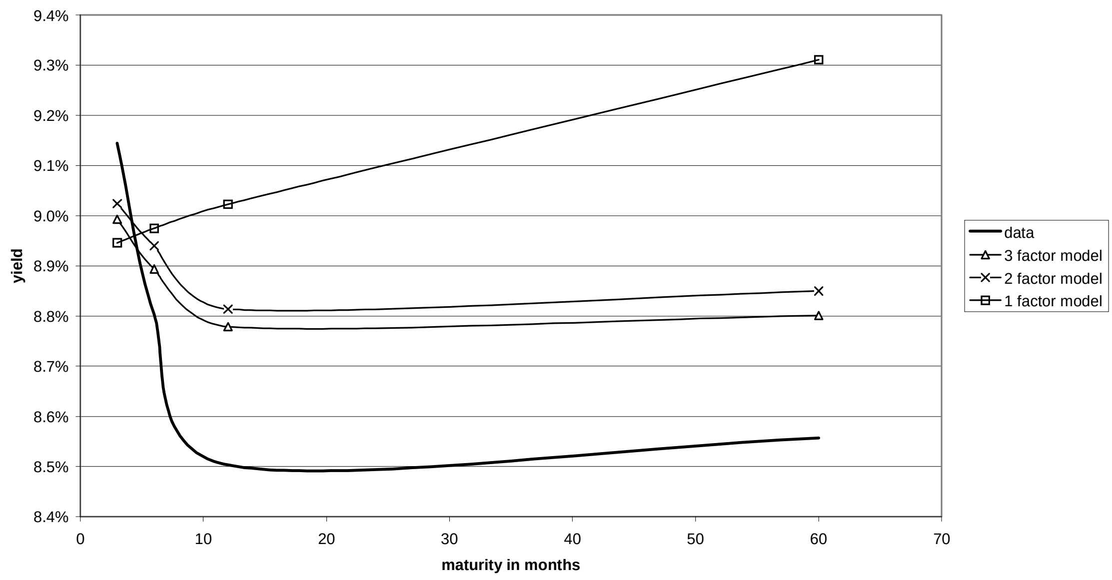

Figure 8b : Germany : steepest inverted term structure (month 72, yield difference -0.01819) and the K factor model fits.

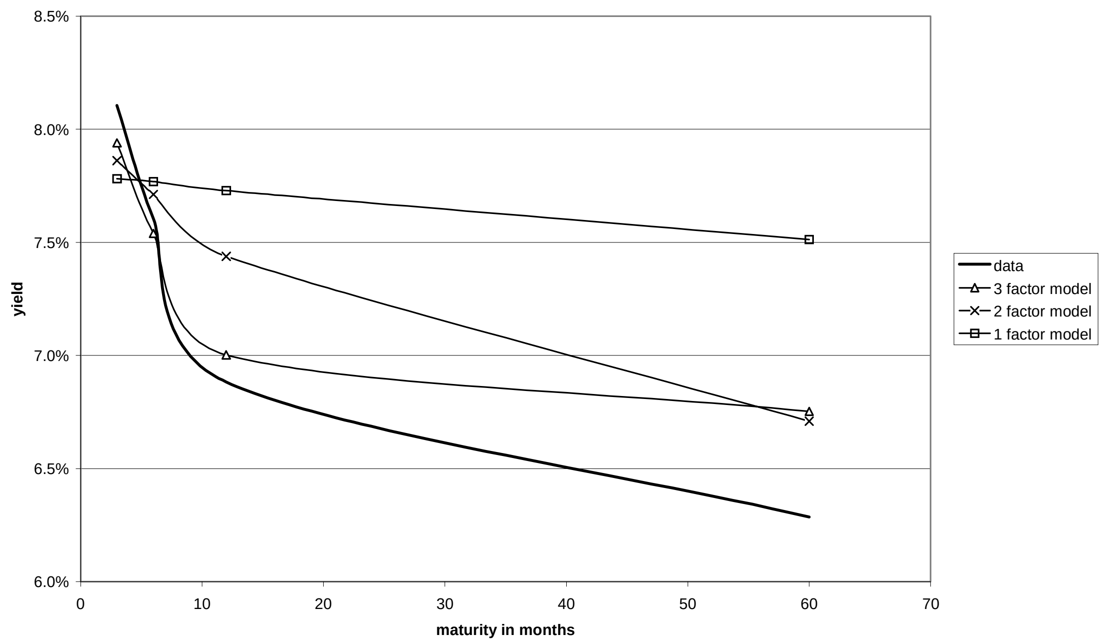

Figure 8c : UK : steepest inverted term structure (month 32, yield difference -0.03231) and the K factor model fits.

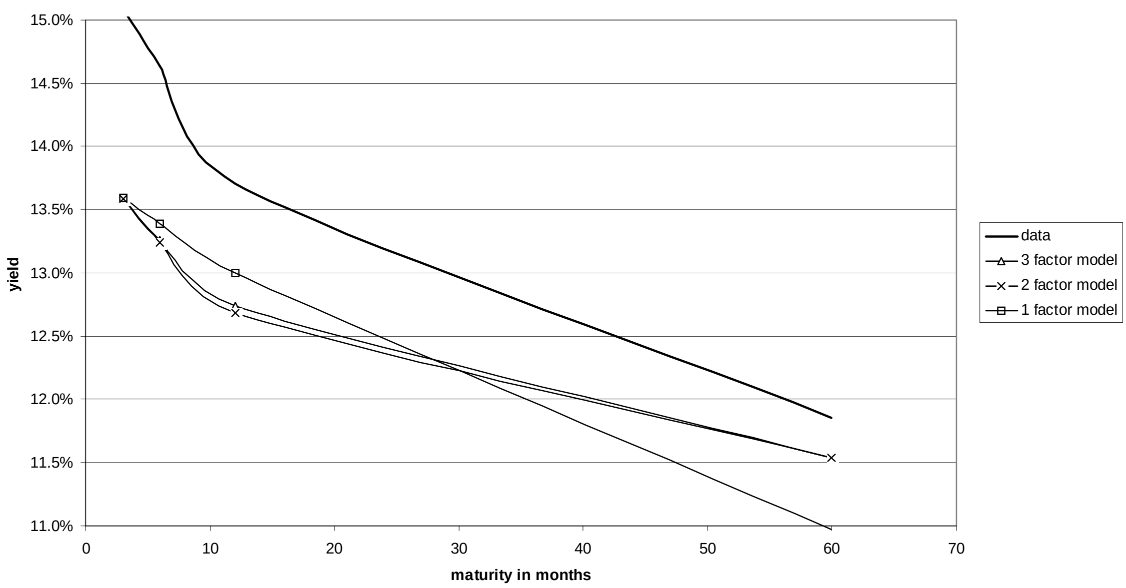

Figure 8a : USA : steepest term structure (month 62, yield difference 0.03184) and the K factor model fit.

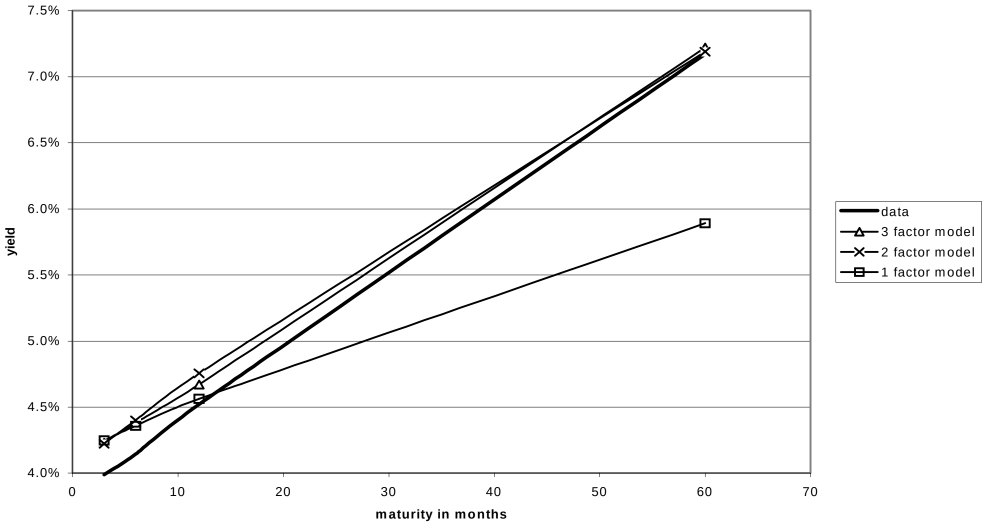

Figure 8b : Germany : steepest term structure (month 112, yield difference 0.02536) and the K factor model fit.

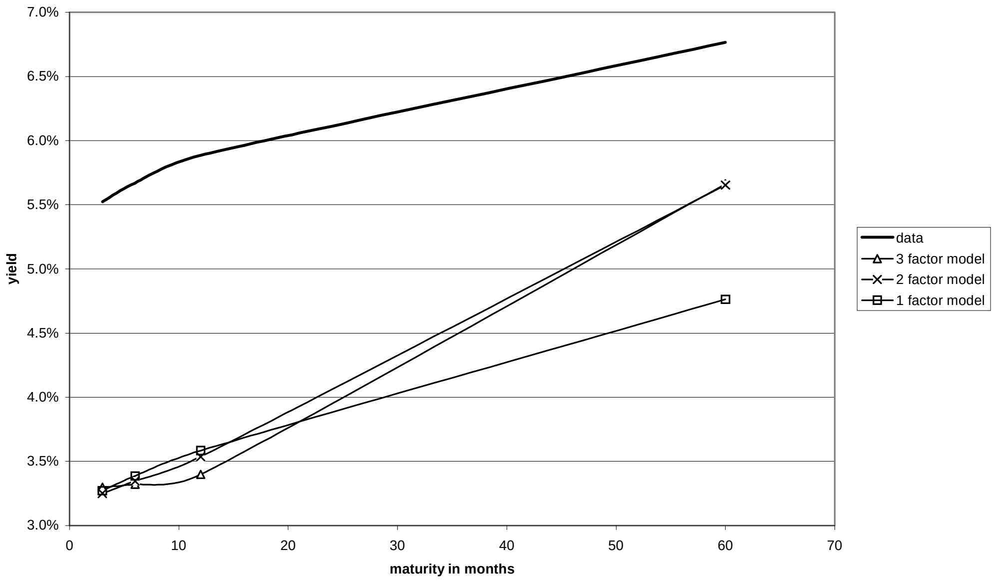

Figure 8c : United Kingdom : steepest term structure (month 88, yield difference 0.0344) and the K factor model fit.

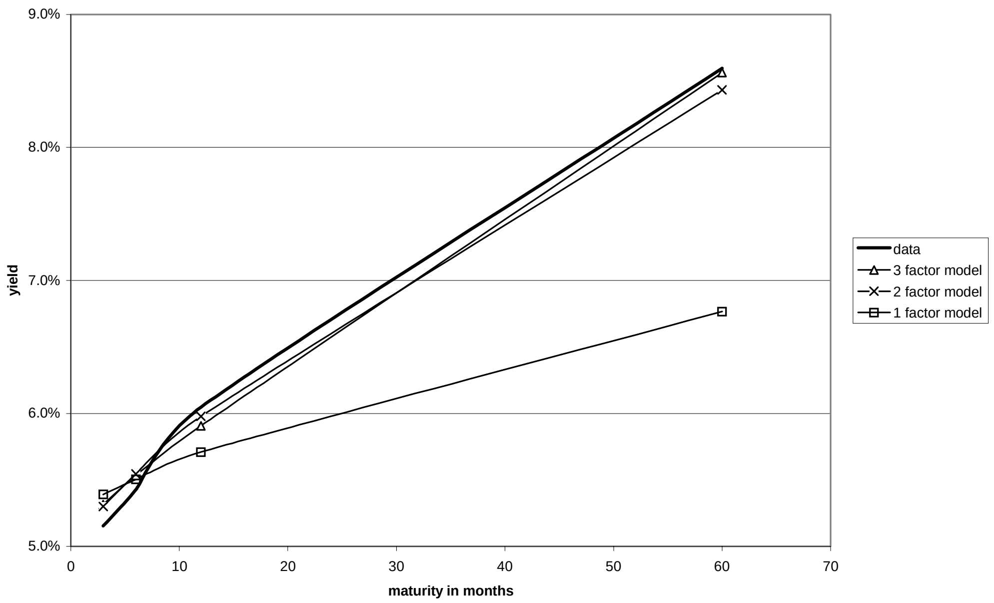

Figure 3 : 5 year yield risk premia for the different countries considered.

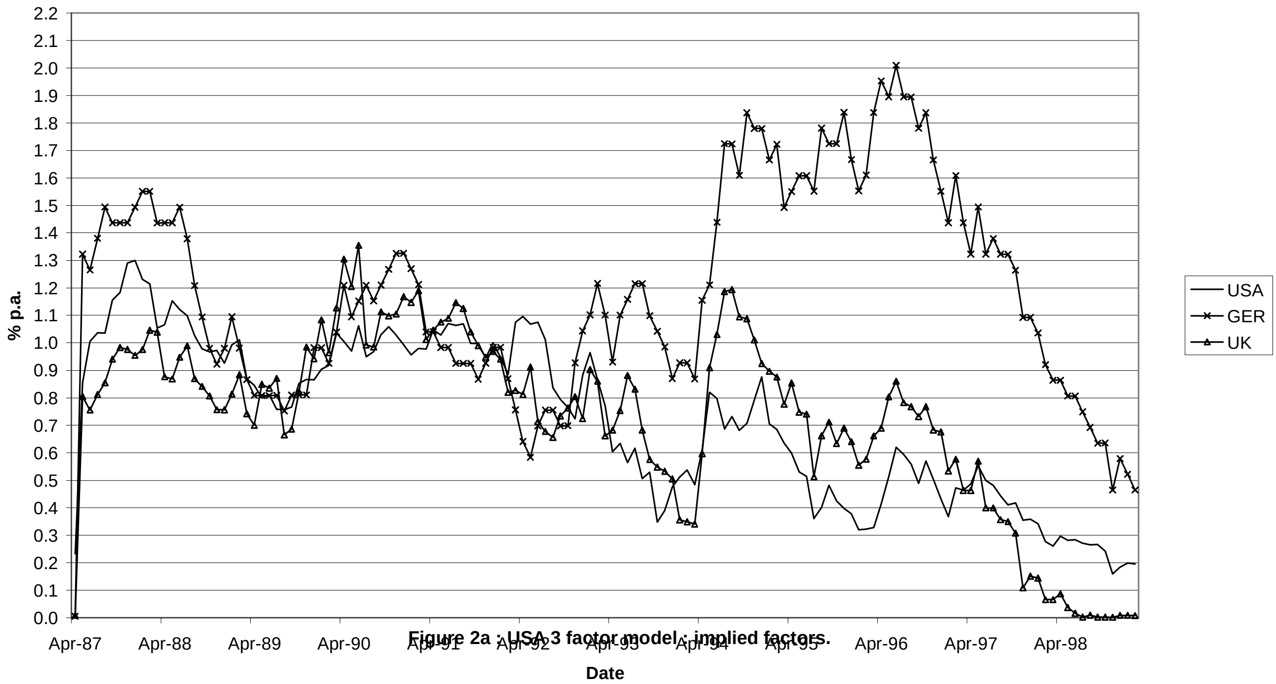

Figure 2a : USA 3 factor model : implied factors.

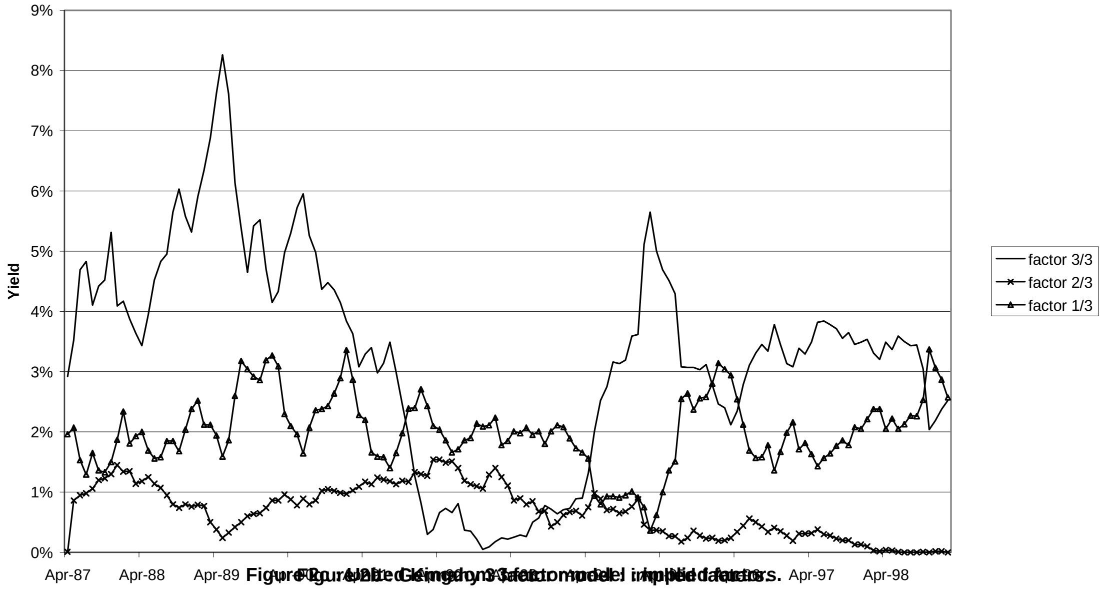

Figure 2b: United Kingdom 3 factor model : implied factors.

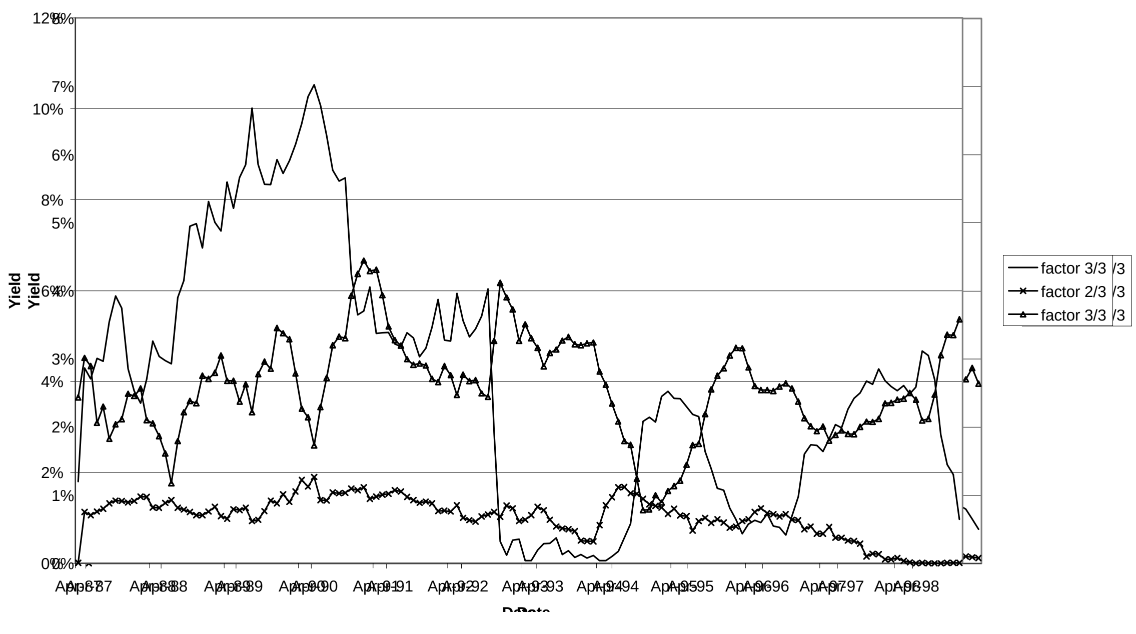

Figure 1a : USA yield data for selected maturities.

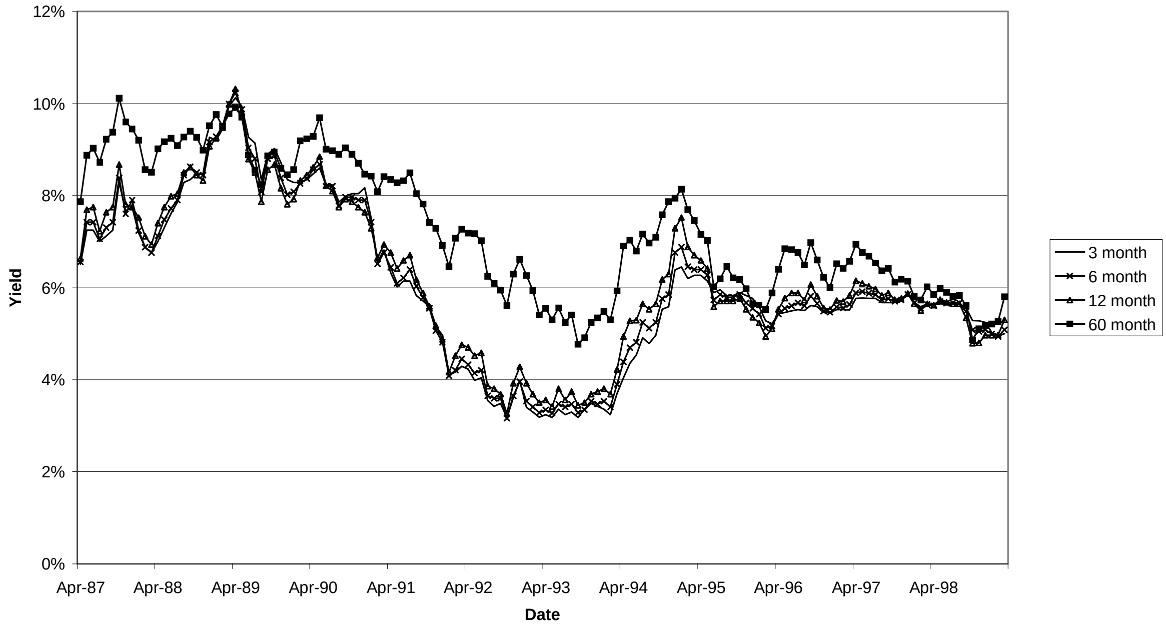

Figure 1b : German yield data for selected maturities.

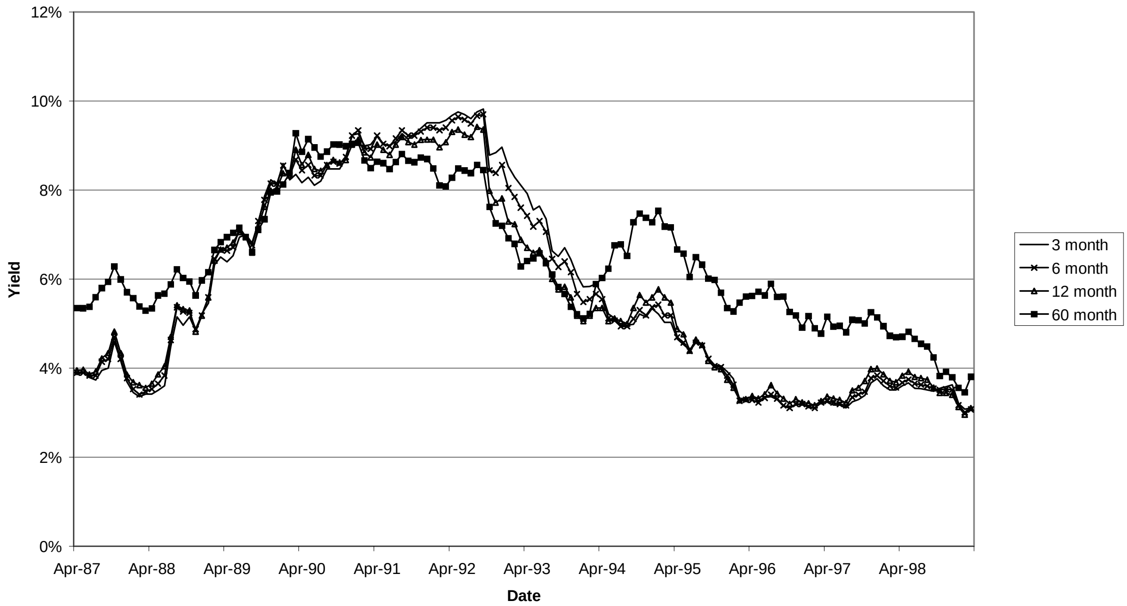

Figure 1c : United Kingdom yield data for selected maturities.